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Algebraic construction of the Darboux matrix revisited

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Abstract
We present algebraic construction of Darboux matrices for 1+1-dimensional integrable systems of nonlinear partial differential equations with a special stress on the nonisospectral case. We discuss different approaches to the Darboux–Bäcklund transformation, based on different λ-dependences of the Darboux matrix: polynomial, sum of partial fractions or the transfer matrix form. We derive symmetric *N*-soliton formulae in the general case. The matrix spectral parameter and dressing actions in loop groups are also discussed. We describe reductions to twisted loop groups, unitary reductions, the matrix Lax pair for the KdV equation and reductions of chiral models (harmonic maps) to $SU(n)$ and to Grassmann spaces. We show that in the KdV case the nilpotent Darboux matrix generates the binary Darboux transformation. The paper is intended as a review of known results (usually presented in a novel context) but some new results are included as well, e.g., general compact formulae for *N*-soliton surfaces and linear and bilinear constraints on the nonisospectral Lax pair matrices which are preserved by Darboux transformations.

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1. Introduction

A (1+1)-dimensional integrable system can be considered as integrability conditions for a *linear problem* (a system of linear partial differential equations defined by two matrices containing the *spectral parameter*), see for instance [56]. The Darboux–Bäcklund transform is a gauge-like transformation (defined by the *Darboux matrix*) which preserves the form of the linear problem [14, 22, 27, 40, 66]. All approaches to the construction of Darboux matrices originate in the dressing method [56, 68, 81, 82].

The paper is intended as a presentation of Darboux–Bäcklund transformations from a unified perspective, first presented in [13, 14]. The construction of the Darboux matrix is divided into two stages. First, we uniquely characterize the considered linear problem in terms of algebraic constraints (the divisor of poles, loop group reductions and other algebraic properties, e.g., linear and bilinear constraints). Then, we construct the Darboux matrix preserving all these constraints. Using general theorems, including those from the present paper, one may construct the Darboux matrix in a way which is almost algorithmic.

The paper is intended as a review of known results but some new results are also included. We discuss in detail elementary Darboux transformation (Darboux matrix which has a single simple zero), symmetric formulae for Darboux matrices and soliton surfaces (in the general case) and loop group reductions for polynomial Darboux matrices. Two examples are discussed in detail: the Korteweg–de Vries equation and chiral models (harmonic maps).

The part which seems to be most original contains the description of linear and bilinear invariants of Darboux transformations. We prove that multilinear constraints introduced in [14] are invariant with respect to the polynomial Darboux transformation (also in the nonisospectral case). Taking them into account we can avoid some cumbersome calculations, our construction assumes a more elegant form and, last but not least, we do not need any assumptions concerning boundary conditions.

Another important aim of this paper is to show similarities and even an equivalence between different algebraic approaches to the construction of the Darboux matrix. This is a novelty in itself because sometimes it is difficult to note connections between different methods. The existing monographs, even the recent ones, focus on a chosen single approach, compare [24, 27, 46, 47, 56, 60].

We consider a nonlinear system of partial differential equations which is equivalent to the compatibility conditions

$$U_{\mu,\nu} - U_{\nu,\mu} + [U_\mu, U_\nu] = 0, \qquad (1 \leqslant \mu < \nu \leqslant m), \tag{1.1}$$

for the following system of linear equations (known as the Lax pair, at least in the case of two independent variables)

$$\Psi_{,\nu} = U_\nu \Psi, \qquad (\nu = 1, \ldots, m), \tag{1.2}$$

where $n \times n$ matrices U_ν depend on $x^1, \ldots, x^m$ and on the so-called spectral parameter λ (and, as usual, $\Psi_{,\nu} = \partial\Psi/\partial x^\nu$, etc). We assume that Ψ is also a matrix (the fundamental solution of the linear system (1.2)). We fix our attention on the case $m = 2$ (although most results hold for any m) and briefly denote by x the set of all variables, i.e., $x = (x^1, \ldots, x^m)$.

1.1. Isospectral and nonisospectral Lax pairs

Let us recall that the most important characteristic of the matrices U_1, U_2 is their dependence on the spectral parameter λ. In the typical case U_ν are rational with respect to λ. Actually we will consider a more general situation. We assume that the Lax pair is rational with respect to λ, and

- 'isospectral case': λ is a constant parameter;
- 'non-isospectral' case:

  $$\lambda_{,\nu} = L_\nu(x, \lambda), \qquad (\nu = 1, \ldots, m), \tag{1.3}$$

  where L_ν are given functions, rational with respect to λ (this case reduces to the isospectral one for $L_\nu(x, \lambda) \equiv 0$).

Remark 1.1. The differential equations (1.3) are of the first order, so their solution $\lambda = \Lambda(x, \zeta)$ depends on a constant of integration ζ which plays the role of the constant spectral parameter.

The solution of the system (1.3) exists provided that compatibility conditions hold, for more details see [14]. In general $\Lambda = \Lambda(x, \zeta)$ is an implicit function, although in many special cases explicit expression for Λ can be found, compare [11, 14, 69].

1.2. The Darboux–Bäcklund transformation

The application of the dressing method to generate new solutions of nonlinear equations 'coded' in (1.1) consists in the following (see [56, 79, 82]). Suppose that we are able to construct a gauge-like transformation $\tilde{\Psi} = D\Psi$ (where $D = D(x, \lambda)$ will be called the Darboux matrix) such that the structure of matrices $\tilde{U}_\nu$,

$$\tilde{U}_\nu = D_{,\nu} D^{-1} + D U_\nu D^{-1}, \qquad (\nu = 1, \ldots, m), \tag{1.4}$$

is identical with the structure of the matrices U_ν. The soliton fields entering U_ν are replaced by some new fields which, obviously, have to satisfy the nonlinear system (1.1) as well.

Remark 1.2. The Darboux transformation should preserve divisors of poles (i.e., poles and their multiplicities) of matrices U_ν. This is the most important structural property of U_ν to be preserved. The second important property is the so-called reduction group, see section 6.

For any pair of solutions of (1.1) one can 'compute' $D := \tilde{\Psi}\Psi^{-1}$. The crucial point is, however, to express D solely by the wavefunction Ψ because only then one can use D to construct new solutions. Such D is known as the Darboux matrix [40, 45, 46]. The Darboux matrix defines an explicit map $S \mapsto S$, where S is the set of solutions of the linear problem (1.2). The construction of the Darboux matrix is based on the important observation:

Remark 1.3. The Darboux matrix can be expressed in an algebraic way by the original wavefunction Ψ.

By the 'original wavefunction' we mean one before the transformation. In fact, it is rather difficult to find special solutions of the linear problem. Usually very limited number of cases is available. However, knowing any solution $\Psi = \Psi(x, \lambda)$ and the Darboux matrix one can generate a sequence of explicit solutions. Starting from the trivial background (x-independent and mutually commuting U_ν) we usually get the so-called soliton solutions.

1.3. Equivalent Darboux matrices

It is quite natural to consider as equivalent Darboux matrices which produce exactly the same transformation (1.4) of matrices U_ν of a given linear problem.

Remark 1.4. The linear problem (1.2) is invariant under transformations $\Psi \mapsto \Psi C_0$ (for any constant nondegenerate matrix $C_0 = C_0(\lambda)$).

Therefore Darboux matrices D and D' are equivalent if there exists a matrix C such that $D\Psi = D'\Psi C$ (for any Ψ). Thus C should commute with Ψ, which, in practice, means that $C = f(\lambda) \in \mathbb{C}$.

Remark 1.5. The matrix $D' = f(\lambda)D$, where f is a complex function of λ only, is equivalent to D.

1.4. Soliton surfaces approach

Given a solution $\Psi = \Psi(x, \lambda)$, where λ depends on x and ζ, we define a new object F by the so-called Sym–Tafel (or Sym) formula:

$$F = \Psi^{-1}\Psi_{,\zeta}. \tag{1.5}$$

If Ψ assumes values in a matrix Lie group G, then (for any fixed ζ) F describes an immersion (a 'soliton surface') into the corresponding Lie algebra [71, 74]. Soliton surfaces are a natural frame to unify a variety of different physical models like soliton fields, strings, vortices, chiral models and spin models [72]. In the framework of the soliton surfaces approach one can reconstruct many integrable cases known from the classical differential geometry [7, 15, 16, 74]. The Darboux–Bäcklund transformation for soliton surfaces reads

$$\tilde{F} = F + \Psi^{-1} D^{-1} D_{,\zeta} \Psi, \tag{1.6}$$

where $D_{,\zeta} = \lambda_{,\zeta} D_{,\lambda}$. The equivalent Darboux matrices yield the same soliton surfaces. Indeed, if we take $D' = f D$, then

$$\tilde{F} = F + \Psi^{-1} D'^{-1} D'_{,\zeta} \Psi = F + \frac{f_{,\zeta}}{f} + \Psi^{-1} D^{-1} D_{,\zeta} \Psi, \tag{1.7}$$

i.e., surfaces corresponding to D and D' differ by the constant $(\ln f)_{,\zeta}$.

In order to illustrate usefulness of the geometric approach we present the following theorem [18].

Theorem 1.6. *We assume that* U_1, U_2 *are linear combinations of* $1, \lambda$ *and* λ^{-1}*, with x-dependent* $su(2)$*-valued coefficients, and* $U_\nu(-\lambda) = E_0 U_\nu(\lambda) E_0^{-1}$ *(where* $E_0 \in su(2)$ *is a constant matrix). Then* F *given by the Sym formula (1.5) is (in the isospectral case) a pseudospherical (i.e., of negative Gaussian curvature) surface immersed in* $su(2) \simeq \mathbb{R}^3$*. In the nonisospectral case the same assumptions yield the so-called Bianchi surfaces.*

We point out that surprisingly few assumptions (restrictions) on the spectral problem leads to the very important class of pseudospherical surfaces. It is easy to assure the preservation of these restrictions by the Darboux transformation.

Darboux transformations usually preserve many other constraints (e.g., linear and bilinear invariants discussed in section 8), which leads to the preservation of some geometric characteristics (e.g., curvature lines) and to a specific choice of coordinates and other auxiliary parameters.

2. Binary Darboux matrix

In this paper by the binary Darboux matrix we mean one pole matrix with non-degenerate normalization

$$D = \mathcal{N} \left(I + \frac{\lambda_1 - \mu_1}{\lambda - \lambda_1} P \right), \qquad P^2 = P, \quad \det \mathcal{N} \neq 0, \tag{2.1}$$

such that its inverse has the same form:

$$D^{-1} = \left(I + \frac{\mu_1 - \lambda_1}{\lambda - \mu_1} P \right) \mathcal{N}^{-1}. \tag{2.2}$$

Here λ_1, μ_1 are complex parameters (which can depend on x in the nonisospectral case), $P = P(x)$ is a projector matrix ($P^2 = P$) and $\mathcal{N} = \mathcal{N}(x)$ is the so-called normalization matrix.

2.1. Binary or elementary?

The name 'binary' for Darboux matrices of the form (2.1) is rather tentative, because binary Darboux transformations were introduced in another context (compare [46, 83]). The 'classical' binary transformation corresponds to the degenerate case of (2.1) when $\mu_1 \to \lambda_1$ (see section 4.4), i.e.,

$$D = \mathcal{N} \left(I + \frac{M}{\lambda - \lambda_1} \right), \qquad D^{-1} = \left(I - \frac{M}{\lambda - \lambda_1} \right) \mathcal{N}^{-1}, \tag{2.3}$$

where $M^2 = 0$ (the so-called nilpotent case, see [14]). Therefore, we use this notion in an extended sense. However, it seems to be compatible with understanding binary Darboux transformation as a composition of an elementary Darboux transformation and a Darboux transformation of the adjoint linear problem [46, 57, 67]. In the case of the Zakharov–Shabat spectral problems (1.2) the adjoint spectral problem is given by

$$-\Phi_{,\nu} = \Phi U_\nu, \tag{2.4}$$

and one can easily check that $\Phi = \Psi^{-1}$ solves the adjoint spectral problem. The general solution of (2.4) is $\Phi = \Psi^{-1}C$, where C is a constant (i.e., x-independent) matrix. We will see in section 5.1 that the binary Darboux matrix can be expressed by a pair of solutions: one solves the spectral problem (1.2) and the second one solves the adjoint problem (2.4).

The matrix (2.1) is equivalent to the linear in λ matrix $\hat{D}$

$$\hat{D} = \mathcal{N}(\lambda - \lambda_1 + (\lambda_1 - \mu_1)P). \tag{2.5}$$

Darboux matrices linear in λ are sometimes referred to as 'elementary', see [60]. Indeed, iterating such transformations we can get any Darboux transformation with nondegenerate normalization. However, we reserve the name 'elementary' for matrices which are not only linear in λ but have a single zero (see section 4), or even a single simple zero. The polynomial form (2.5) of the binary Darboux matrix has two zeros: λ_1, μ_1. The sum of their multiplicities is n. Therefore these zeros are simple only in the case $n = 2$.

2.2. *Sufficient conditions for the projector*

Assuming that U_ν are regular (holomorphic) at $\lambda = \lambda_1$ and $\lambda = \mu_1$, and demanding that $\tilde{U}_\nu$ (expressed by (1.4)) have no poles at $\lambda = \lambda_1$ and $\lambda = \mu_1$ as well, we get the following conditions (for vanishing corresponding residues), compare [14]:

$$\begin{aligned} &P \circ (-\partial_\nu + U_\nu(\lambda_1)) \circ (I - P) = 0,\\ &(I - P) \circ (-\partial_\nu + U_\nu(\mu_1)) \circ P = 0, \end{aligned} \tag{2.6}$$

$$\lambda_{1,\nu} = L_\nu(x, \lambda_1), \qquad \mu_{1,\nu} = L_\nu(x, \mu_1), \tag{2.7}$$

where the circles mean composition of linear operators and L_ν are defined by (1.3). Note that for any operators A, B we have

$$A \circ B = 0 \quad \Rightarrow \quad \operatorname{im} B \subset \ker A. \tag{2.8}$$

Indeed, $(A \circ B)\varphi = 0$ for any vector φ, i.e. $A(B\varphi) = 0$, which means exactly that $B\varphi \in \ker A$. On the other hand, any element of $\operatorname{im} B$ is of the form $B\varphi$.

Remark 2.1. The assumption that U_ν are regular at $\lambda = \lambda_1$ and $\lambda = \mu_1$ (assumed throughout this paper) is essential. Relaxing this requirement we can get solutions different from those obtained by the standard Darboux–Bäcklund transformation. Solutions of this kind (*unitons*) have been found in the case of harmonic maps into Lie groups [77], see also [27].

If the system (1.3) has the general solution $\lambda = \Lambda(x, \zeta)$, then equations (2.7) can be solved in terms of the function Λ:

$$\lambda_1 = \Lambda(x, \zeta_1), \qquad \mu_1 = \Lambda(x, \zeta_1'), \tag{2.9}$$

where ζ_1, ζ_1' are constant parameters, compare [4, 51]. Taking into account $\ker P = \operatorname{im}(I - P)$, we easily show that the system (2.6) is equivalent to

$$\begin{aligned} &(-\partial_\nu + U_\nu(\lambda_1)) \ker P \subset \ker P,\\ &(-\partial_\nu + U_\nu(\mu_1)) \operatorname{im} P \subset \operatorname{im} P. \end{aligned} \tag{2.10}$$

Now we easily see that the conditions (2.10) are satisfied by the projector defined by the Zakharov–Shabat formulae (compare [56, 79, 82])

$$\ker P = \Psi(\lambda_1)V_{\text{ker}}, \quad \operatorname{im} P = \Psi(\mu_1)V_{\text{im}}, \tag{2.11}$$

where $\Psi(\lambda_1) = \Psi(x, \lambda_1)$, $\Psi(\mu_1) = \Psi(x, \mu_1)$ and V_{ker} and V_{im} are constant vector spaces such that $V_{\text{ker}} \oplus V_{\text{im}} = V$. Indeed, in this case, by virtue of (1.2), the left-hand sides of (2.10) are simply equal to zero.

Taking into account that any projector P can be expressed explicitly by its kernel and image, $P = (\operatorname{im} P, 0)(\operatorname{im} P, \ker P)^{-1}$, we can summarize the above discussion as follows.

Proposition 2.2. *The transformation (1.4) with D given by (2.1), where*

$$P = (\Psi(\mu_1)V_{\text{im}}, 0)(\Psi(\mu_1)V_{\text{im}}, \Psi(\lambda_1)V_{\text{ker}})^{-1}, \tag{2.12}$$

preserves the divisors of poles of matrices U_ν.

The formula (2.12) yields a sufficient condition for P to generate the Darboux matrix. It is interesting to find also necessary conditions. Therefore, we will try to obtain the most general solution to (2.10).

2.3. The general form of the binary Darboux matrix

It is convenient to represent vector spaces in a matrix form. Namely, if $w_1, \ldots, w_k$ span a vector space V, then we can identify V with the matrix

$$V = (w_1, \ldots, w_k). \tag{2.13}$$

This matrix has k columns $(w_1, \ldots, w_k)$ and n rows $(n = \dim V)$.

Note that because of the freedom in choosing a basis in the vector space there are many matrices representing the same vector space. If a_{ij} are coefficients of a $k\times k$ non-degenerate matrix A, then the vectors

$$w'_j = \sum_{i=1}^{k} w_i a_{ij},$$

form another basis in V which can be represented by the matrix

$$V' = (w'_1, \ldots, w'_k) = VA.$$

The matrices V and V' (for any non-degenerate A) represent the same vector space and, in this context, are considered as equivalent.

The space of k-dimensional subspaces of an n-dimensional vector space over $\mathbb{C}$ is known as Grassmannian $G_{k,n}(\mathbb{C})$ (of course, considering real vector spaces we have real Grassmannian $G_{k,n}(\mathbb{R})$). The elements of the Grassmannian are classes of equivalence of $k\times n$ matrices with respect to the equivalence relation: $V \simeq V'$ if there exists $k\times k$ matrix A $(\det A \neq 0)$ such that $V' = VA$.

Therefore using the same notation for the vector space and the matrix representing it, one should remember about this equivalence. In particular, in order to show that some vector spaces W and V are identical, one has to consider the equation $W = VA$ with an arbitrary non-degenerate A. In the similar way one can check whether W is a subspace of V (of course a necessary requirement is $\dim V \geqslant \dim W$).

Proposition 2.3. *Let* V, W *be vector spaces,* $k' = \dim W \leqslant \dim V = k$. *Then* $W \subset V$ *if and only if there exists* $k\times k$ *matrix B such that* $W = VB$.

Proof. If $W \subset V$, then there exists a basis of V such that its first k' vectors span W. We represent V by vectors of this basis, i.e., we choose A such that $w'_1, \ldots, w'_{k'}$ span W. Finally, we put $B = A\,\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$. □

Note that formally W and V belong (in general) to different Grassmannians. But if $\det B = 0$, then the columns of VB are linearly dependent and VB can be treated as an element of a Grassmannian of lower dimension.

We proceed to solving the system (2.10). Assuming $\det \Psi(\lambda_1) \neq 0$ and $\det \Psi(\mu_1) \neq 0$, we can always put $\ker P = \Psi(\lambda_1)V$, $\mathrm{im}\, P = \Psi(\mu_1)W$, where W, V are some vector spaces (in general x-dependent). Substituting into (2.10) we have $\Psi(\lambda_1)V_{,\nu} \subset \Psi(\lambda_1)V$, $\Psi(\mu_1)W_{,\nu} \subset \Psi(\mu_1)W$. Hence,

$$V_{,\nu} \subset V, \qquad W_{,\nu} \subset W.$$

By proposition 2.3, we rewrite $V_{,\nu} \subset V$ as $V_{,\nu} = VB_\nu$ for some B_ν (which have to satisfy appropriate compatibility conditions) and analogous equations for W. Taking into account the freedom of changing the basis when changing x: $V' = VA$ ($\det A \neq 0$), we obtain

$$(V'A^{-1})_{,\nu} = V'A^{-1}B_\nu.$$

Therefore, choosing A such that

$$(A^{-1})_{,\nu} = A^{-1}B_\nu, \tag{2.14}$$

we obtain $V'_{,\nu} = 0$, i.e. there exists an x-independent basis in V (the same conclusion holds for W). The solution to (2.14) exists because B_ν satisfy the compatibility conditions mentioned above. Thus we have shown that formulae (2.11) give the most general solution to (2.10).

3. Polynomial Darboux matrices: general case

In this paper we consider only rational Darboux matrices ($n \times n$ matrices with coefficients which are rational functions of λ).

Remark 3.1. In the isospectral case, every rational Darboux matrix is equivalent to a polynomial Darboux matrix

$$\hat{D} = \sum_{k=0}^{N} T_k(x)\lambda^{N-k}. \tag{3.1}$$

Indeed, it is enough to multiply given D by the least common multiple of all denominators. The obtained polynomial will be denoted by $\hat{D}$.

Another equivalent form of D is a polynomial in λ^{-1}, obtained from $\hat{D}(\lambda)$ by dividing it by λ^N. In some cases this polynomial is more convenient than $\hat{D}$ because it is analytic at $\lambda = \infty$.

In the nonisospectral case the least common multiple of all denominators depends on x. Therefore, any rational Darboux matrix is equivalent to some polynomial matrix up to a scalar x-dependent factor.

3.1. The determinant of the Darboux matrix

The trace of a quadratic matrix is defined as the sum of diagonal elements of this matrix. Both the trace and the determinant are invariant with respect to similarity transformations: $\mathrm{Tr}(BAB^{-1}) = \mathrm{Tr}\, A$, $\det(BAB^{-1}) = \det A$.

Theorem 3.2 (Liouville)**.** *If* $\Psi_{,\nu} = U_\nu \Psi$, *where* ν *is fixed and* $U_\nu = U_\nu(x)$ *is given, then*

$$(\det \Psi)_{,\nu} = \operatorname{Tr} U_\nu \det \Psi. \tag{3.2}$$

This theorem is well known as the Liouville theorem on Wronskians, see, for instance, [2].

Applying the Liouville theorem to the Darboux transform $\tilde{\Psi} \equiv D\Psi$ we get $(\det D \det \Psi)_{,\nu} = \operatorname{Tr} \tilde{U}_\nu \det D \det \Psi$. Hence, using once more (3.2), we obtain

$$\frac{(\det D)_{,\nu}}{\det D} = \operatorname{Tr}(\tilde{U}_\nu) - \operatorname{Tr}(U_\nu). \tag{3.3}$$

Remark 3.3. We usually consider traceless linear problems ($\operatorname{Tr} U_\nu = 0$ for $\nu = 1, \ldots, m$). In such case $\det D$ has to be constant (i.e., $\det D$ does not depend on x). Therefore, in the isospectral (and traceless) case $\det D$ can depend only on λ and all its zeros are constants.

In the nonisospectral case the situation is more complicated because λ depends on x. However, it is still possible to obtain a strong general result characterizing zeros of $\det D$.

Theorem 3.4. *We consider a polynomial Darboux matrix* $\hat{D}$ *for a nonisospectral linear problem (1.2) with* λ *satisfying (1.3). If* $\det \hat{D}(\lambda_k) = 0$ *and matrices* U_ν *are regular at* λ_k, *then*

$$\lambda_{k,\nu} = L_\nu(x, \lambda_k), \tag{3.4}$$

i.e., $\lambda_k = \Lambda(x, \zeta_k)$, *where* $\zeta_k = \text{const}$.

Proof. The determinant of the polynomial $\hat{D}(\lambda)$ has a finite number of roots (x-dependent, in general). We denote them by $\lambda_k, k = 1, \ldots, K$, and their multiplicities by m_k. Note that $m_1 + m_2 + \cdots + m_K = nN$, where N is the degree of the polynomial $\hat{D}(\lambda)$ and n is the order of the matrix $\hat{D}$. Thus

$$\det \hat{D}(\lambda) = h \prod_{k=1}^{K} (\lambda - \lambda_k)^{m_k}, \tag{3.5}$$

where $h = h(x)$ and $\lambda_k = \lambda_k(x)$. Taking into account (1.3) we compute

$$\frac{(\det D)_{,\nu}}{\det D} = \frac{h_{,\nu}}{h} + \sum_{k=1}^{K} \left(m_k \frac{L_\nu(x, \lambda) - \lambda_{k,\nu}}{\lambda - \lambda_k} \right). \tag{3.6}$$

Equation (3.3) with U_ν regular at λ_k implies that the right-hand side of (3.6) should have no poles. Therefore residues of (3.6) at $\lambda = \lambda_k$ vanish, which implies (3.4). The x-dependence of λ_k follows from remark 1.1. □

The regularity of U_ν at $\lambda = \lambda_k$ is assumed throughout this paper. If we allow that some λ_k coincides with a singularity of U_ν, then the x-dependence of λ_k in principle can be different from (3.4) and we get an additional freedom.

3.2. Neugebauer's approach

A simple but quite general method to construct polynomial Darboux–Bäcklund transformations has been proposed by Neugebauer and his collaborators [47, 51, 52], see also [34, 60]. We are going to find conditions on polynomial $\hat{D}$ implying that divisors of poles of $\tilde{U}_\nu$ and U_ν coincide (compare remark 1.2). From $\tilde{\Psi}_{,\nu} = \tilde{U}_\nu \tilde{\Psi}$ we get

$$\tilde{U}_\nu = \frac{\tilde{\Psi}_{,\nu}(\lambda)\tilde{\Psi}^c(\lambda)}{\det \tilde{\Psi}(\lambda)} = \frac{1}{\det \hat{D}} (\hat{D}_{,\nu}\hat{D}^c + \hat{D} U_\nu \hat{D}^c), \tag{3.7}$$

where by $\hat{D}^c$ we denote the matrix of cofactors of $\hat{D}$. Obviously $\hat{D}^c$ is also a polynomial in λ.

If U_ν are rational functions of λ, then $\tilde{U}_\nu$ given by (3.7) are rational as well (because $\hat{D}$ and $\hat{D}^{-1}$ are rational). Therefore the only candidates for poles of $\tilde{U}_\nu$ are poles of U_ν and zeros of $\det \hat{D}$ (i.e., λ_k). The necessary condition for the regularity of $\tilde{U}_\nu$ at $\lambda = \lambda_k$ is

$$\tilde{\Psi}_{,\nu}(\lambda_k)\tilde{\Psi}^c(\lambda_k) = 0. \tag{3.8}$$

If λ_k is a simple zero of $\det \hat{D}(\lambda)$, then the condition (3.8) is also sufficient.

Following [47], we will find another, more constructive, characterization of the condition (3.8). If $\det \hat{D}(\lambda_k) = 0$, then we have also

$$\det \tilde{\Psi}(\lambda_k) = 0 \tag{3.9}$$

(because $\tilde{\Psi}(\lambda) = \hat{D}(\lambda)\Psi(\lambda)$). We assume that the function $\Psi(\lambda)$ (known as a 'background solution' or a 'seed solution') is non-degenerate at $\lambda = \lambda_k$.

As a consequence of (3.9), the equation $\tilde{\Psi}(\lambda_k)p_k = 0$ has a non-zero solution $p_k \in \mathbb{C}^n$ (where, in principle, p_k can depend on x). Then, we compute

$$\tilde{\Psi}_{,\nu}(\lambda_k)p_k = \tilde{U}_\nu(\lambda_k)\tilde{\Psi}(\lambda_k)p_k = 0,$$

where we took into account that $\tilde{\Psi}(\lambda)$ satisfies (1.2). Thus we have

$$\tilde{\Psi}(\lambda_k)p_k = \tilde{\Psi}_{,\nu}(\lambda_k)p_k = 0, \tag{3.10}$$

which implies (3.8), as one can see from the following fact of linear algebra ([47], see also [34]).

Lemma 3.5. *Let us consider two degenerate matrices X and Y. Suppose that there exists a vector p such that $Xp = 0$ and $Yp = 0$. Then $YX^c = 0$.*

Proof. Let us perform computations in a basis $(e_1, \ldots, e_n)$ such that $e_1 \equiv p$. Then all elements of the first column of matrices X, Y are equal to zero. Thus, using the definition of the cofactor, we easily see that the rows of Y^c (except the first row) have all entries equal to zero. Hence, XY^c obviously yields zero. □

Lemma 3.6. *The vector p_k such that $\tilde{\Psi}(\lambda_k)p_k = 0$ is defined up to a scalar factor. If λ_k is a simple zero, then we can choose this multiplier in such a way that $p_k = \text{const}$.*

Proof. We differentiate the equation defining p_k: $\tilde{\Psi}_{,\nu}(\lambda_k)p_k + \tilde{\Psi}(\lambda_k)p_{k,\nu} = 0$. Hence, $\tilde{\Psi}(\lambda_k)p_{k,\nu} = 0$, which means that $p_{k,\nu}$ is proportional to p_k (provided that λ_k is a simple zero of $\det \hat{D}(\lambda)$). Thus $p_{k,\nu} = f_{k\nu}p_k$, where $f_{k\nu}$ are some scalar functions. From the identity $p_{k,\nu\mu} \equiv p_{k,\mu\nu}$ it follows that $f_{k\nu,\mu} = f_{k\mu,\nu}$. Therefore, there exists φ_k such that $f_{k\nu} = \varphi_{k,\nu}$. Hence, $p_k \mathrm{e}^{-\varphi_k}$ does not depend on x. □

Corollary 3.7. *Polynomial Darboux matrix (3.1) can be constructed as follows. In the isospectral case we choose Nn pairwise different complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_{Nn}$ and Nn constant $\mathbb{C}^n$-vectors $p_1, p_2, \ldots, p_{Nn}$. We also choose the matrix T_0 ('normalization matrix'), $\det T_0 \neq 0$. Matrix coefficients $T_1, \ldots, T_N$ are computed from*

$$\hat{D}(\lambda_k)\Psi(\lambda_k)p_k = 0, \qquad (k = 1, \ldots, nN), \tag{3.11}$$

where $\Psi(\lambda)$ is given ('seed solution'). In the nonisospectral case we choose constants $\zeta_1, \ldots, \zeta_{Nn}$ and use (3.4).

For a fixed k equation (3.11) consists of n scalar equations. Thus we have a system of n^2N equations for N unknown matrices $n \times n$. In the generic case such a system should have a unique solution.

The freedom in choosing T_0 corresponds to a gauge transformation. Note that an identical situation occurs in the case of the binary Darboux matrix, where $\mathcal{N}$ is, in general, undetermined. Usually it is sufficient to put $T_0 = I$ ('canonical normalization'). If this choice leads to a contradiction (i.e., the Darboux matrix with the canonical normalization does not exist), then we may relax this assumption and search for Darboux matrices with more general normalization.

The case $\det T_0 = 0$ can be treated in a similar way but with one exception: the total number of zeros is smaller than Nn. As an example of such a situation we will present elementary Darboux matrices, see section 4.

3.3. Explicit multisoliton formulae

Let us introduce the notation

$$\varphi_k := \Psi(\lambda_k) p_k, \tag{3.12}$$

where $\varphi_k \in \mathbb{R}^n$ are column vectors. We assume $\det T_0 \neq 0$ and denote

$$\theta_j := T_0^{-1} T_j \qquad (j = 1, \ldots, N), \tag{3.13}$$

where T_j are defined by (3.1). Equations (3.11) read

$$\left(\lambda_k^N + \sum_{j=1}^{N} \lambda_k^{N-j} \theta_j \right) \varphi_k = 0 \qquad (k = 1, \ldots, M), \tag{3.14}$$

where $M = nN$. After the transposition we get

$$\sum_{j=1}^{N} \varphi_k^T \theta_j^T \lambda_k^{N-j} = -\varphi_k^T \lambda_k^N. \tag{3.15}$$

It is convenient to solve these equations in the matrix form:

$$\begin{pmatrix} \theta_1^T \\ \theta_2^T \\ \cdots \\ \theta_N^T \end{pmatrix} = - \begin{pmatrix} \lambda_1^{N-1} \varphi_1^T & \ldots & \lambda_1 \varphi_1^T & \varphi_1^T \\ \lambda_2^{N-1} \varphi_2^T & \ldots & \lambda_2 \varphi_2^T & \varphi_2^T \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_M^{N-1} \varphi_M^T & \ldots & \lambda_M \varphi_M^T & \varphi_M^T \end{pmatrix}^{-1} \begin{pmatrix} \varphi_1^T \lambda_1^N \\ \varphi_2^T \lambda_2^N \\ \cdots \\ \varphi_M^T \lambda_M^N \end{pmatrix}. \tag{3.16}$$

Usually, in practical applications, one uses Cramer's rule to express θ_k in terms of determinants, compare [52, 58, 70].

Having coefficients T_k we can apply the Darboux transformation to Lax pairs of prescribed form. As an illustrative example we present the simplest but very important case (linear in λ):

$$U_1 = u_0 \lambda + u_1. \tag{3.17}$$

Equation (1.4) for $\nu = 1$, i.e., $\tilde{U}_1 D = D U_1 + D_{,1}$, yields

$$(\tilde{u}_0 \lambda + \tilde{u}_1) \sum_{k=0}^{N} \lambda^{N-k} T_k = \sum_{k=0}^{N} \lambda^{N-k} T_k (u_0 \lambda + u_1) + \sum_{k=0}^{N} \lambda^{N-k} T_{k,1}. \tag{3.18}$$

Considering coefficients by λ^{N+1} and λ^N, we get explicit formulae for the transformed fields $\tilde{u}_1$ and $\tilde{u}_0$:

$$\tilde{u}_0 = T_0 u_0 T_0^{-1}, \qquad \tilde{u}_1 = T_0 u_1 T_0^{-1} + \left[T_1 T_0^{-1}, \tilde{u}_0 \right] + T_{0,1} T_0^{-1}. \tag{3.19}$$

In the classical AKNS case $u_0 = \mathrm{i}\sigma_3 \equiv \mathrm{diag}(\mathrm{i}, -\mathrm{i})$ and it is sufficient to take the canonical normalization $T_0 = I$. Therefore, we get

$$\tilde{u}_0 = u_0, \qquad \tilde{u}_1 = u_1 + [T_1, \mathrm{i}\sigma_3], \tag{3.20}$$

where $T_1 = \theta_1$ can be explicitly computed from (3.16), compare [52].

4. Elementary Darboux matrix

The elementary Darboux matrix is linear in λ and its determinant has just a single simple zero. This case is mentioned by Its [35] and discussed in more detail in, for instance, [24, 38]. An obvious way to produce matrices of this type is to take matrices with a single entry linear in λ and all other entries λ-independent. In this paper we confine ourselves to elementary Darboux matrices for $n = 2$. They can be represented in the form

$$D = \mathcal{N}\begin{pmatrix} \lambda - \lambda_1 & 0 \\ -\alpha & 1 \end{pmatrix}\mathcal{M} \tag{4.1}$$

where $\mathcal{N}$, $\mathcal{M}$ do not depend on λ. As a simple exercise (compare corollary 3.7) we can express the coefficient α by Ψ evaluated at λ_1, namely:

$$\alpha = \frac{\eta_1}{\xi_1}, \qquad \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = \mathcal{M}\Psi(\lambda_1)p_1, \tag{4.2}$$

where p_1 is a constant vector.

4.1. Binary Darboux matrix as a superposition of elementary transformations

Theorem 4.1. *In the case $n = 2$ any binary Darboux transformation is a superposition of two elementary Darboux transformations.*

Proof. We will show that

$$D = \mathcal{N}_2\begin{pmatrix} 1 & -\beta \\ 0 & \lambda - \lambda_2 \end{pmatrix}\mathcal{N}_1^{-1}\mathcal{N}_1\begin{pmatrix} \lambda - \lambda_1 & 0 \\ -\alpha & 1 \end{pmatrix}\mathcal{M} \tag{4.3}$$

is a binary Darboux matrix ($\mathcal{N}_1, \mathcal{N}_2, \mathcal{M}$ are non-degenerate matrices which do not depend on λ). First, performing the multiplication in (4.3), we get

$$D = \mathcal{N}(\lambda - \lambda_1 + (\lambda_1 - \lambda_2)P), \tag{4.4}$$

where

$$\begin{aligned} \mathcal{N} &= \mathcal{N}_2\begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix}\mathcal{M}, \\ P &= \frac{1}{\Delta\lambda}\mathcal{M}^{-1}\begin{pmatrix} \alpha\beta & -\beta \\ \alpha(\alpha\beta - \Delta\lambda) & \Delta\lambda - \alpha\beta \end{pmatrix}\mathcal{M}, \end{aligned} \tag{4.5}$$

and $\Delta\lambda = \lambda_1 - \lambda_2$. Then, we easily check that $P^2 = P$.

The coefficients α, β can be expressed by Ψ evaluated at λ_1, λ_2. Indeed, denoting

$$\Psi(\lambda_k)p_k = \begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix}, \tag{4.6}$$

and using equations (3.11), we obtain

$$\alpha = \frac{\eta_1}{\xi_1}, \qquad \beta = \frac{\xi_1\xi_2\Delta\lambda}{\xi_2\eta_1 - \eta_2\xi_1}. \tag{4.7}$$

The projector P reads

$$P = \frac{1}{\xi_1\eta_2 - \eta_1\xi_2}\mathcal{M}^{-1}\begin{pmatrix} -\eta_1\xi_2 & \xi_1\xi_2 \\ -\eta_1\eta_2 & \eta_2\xi_1 \end{pmatrix}\mathcal{M}. \tag{4.8}$$

If $\mathcal{M} = I$, then $\Psi(\lambda_1)p_1 \in \ker P$ and $\Psi(\lambda_2)p_2 \in \operatorname{im} P$. Therefore, the binary Darboux matrix with P given by (4.8) is a superposition of elementary transformations (4.3) with $\mathcal{M} = I$ and α, β given by (4.7). □

4.2. KdV equation

The Darboux transformation for the famous Korteweg–de Vries equation is almost always presented in the scalar case, see [46]. The matrix approach is less convenient. However, having in mind a pedagogical motivation, we are going to show in detail that the matrix construction works also in that case. It is interesting that in this paper we do not need the 'KdV reality condition' (usually used in earlier papers, compare [14, 27, 76]).

The standard scalar Lax pair for KdV equation consists of the Sturm–Liouville–Schrödinger spectral problem and the second equation defining the time evolution of the wavefunction:

$$-\psi_{,11} + u\psi = \lambda\psi, \qquad \psi_{,2} = -4\psi_{,111} + 6u\psi_{,1} + 3u_{,1}\psi. \tag{4.9}$$

The compatibility conditions $\psi_{,112} = \psi_{,211}$ yield the KdV equation

$$u_{,2} - 6uu_{,1} + u_{,111} = 0. \tag{4.10}$$

The Lax pair (4.9) can be transformed, in a standard way, to the matrix form

$$\begin{aligned} \Psi_{,1} &= \begin{pmatrix} 0 & 1 \\ u-\lambda & 0 \end{pmatrix} \Psi, \\ \Psi_{,2} &= \begin{pmatrix} -u_{,1} & 2u+4\lambda \\ -4\lambda^2 + 2u\lambda + 2u^2 - u_{,11} & u_{,1} \end{pmatrix} \Psi, \end{aligned} \tag{4.11}$$

where

$$\Psi = (\vec{\psi}, \vec{\phi}), \qquad \vec{\psi} = \begin{pmatrix} \psi \\ \psi_{,1} \end{pmatrix}, \qquad \vec{\phi} = \begin{pmatrix} \phi \\ \phi_{,1} \end{pmatrix}, \tag{4.12}$$

and ψ, ϕ are linearly independent solutions of (4.9).

Lemma 4.2. *Suppose that*

$$U = \begin{pmatrix} 0 & 1 \\ u-\lambda & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & 4\lambda \\ 2\lambda u - 4\lambda^2 & 0 \end{pmatrix} + \begin{pmatrix} -a & b \\ c & a \end{pmatrix}, \tag{4.13}$$

where u, a, b, c do not depend on λ. Then, the compatibility conditions $U_{,2} - V_{,1} + [U, V] = 0$ uniquely yield

$$a = u_{,1}, \qquad b = 2u, \qquad c = 2u^2 - u_{,11}, \tag{4.14}$$

i.e., U, V given by (4.13) are identical with the Lax pair (4.11) for the KdV equation.

Proof is straightforward. Compatibility conditions reduce to (4.10) and (4.14). □

4.3. Elementary Darboux matrix and the classical Darboux transformation

We will compute the action of the elementary Darboux transformation in the KdV case, compare [27]. We assume

$$D = \mathcal{N} \begin{pmatrix} \lambda - \lambda_1 & 0 \\ -\alpha & 1 \end{pmatrix}, \tag{4.15}$$

where $\mathcal{N}$ ($\det \mathcal{N} \neq 0$) does not depend on λ and α is a function to be expressed by $\Psi(\lambda_1)$, namely

$$D(\lambda_1)\Psi(\lambda_1)p_1 = 0, \tag{4.16}$$

where p_1 is a constant vector. We denote

$$\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = \Psi(\lambda_1)p_1. \tag{4.17}$$

The constraint (4.16) (with D given by (4.15)) is equivalent to

$$\alpha = \frac{\eta_1}{\xi_1} = \frac{\hat{\psi}_{1,1}}{\hat{\psi}_1}, \tag{4.18}$$

where $\hat{\psi}_1$ satisfies (4.9) with $\lambda = \lambda_1$ (i.e., $\hat{\psi}_1$ is a linear combination of ψ_1 and ϕ_1). The function α satisfies the following system of Riccati equations:

$$\begin{aligned} \alpha_{,1} &= u - \lambda_1 - \alpha^2, \\ \alpha_{,2} &= \left(2u^2 - u_{,11} + 2u\lambda_1 - 4\lambda_1^2\right) + 2u_{,1}\alpha - (2u + 4\lambda_1)\alpha^2, \end{aligned} \tag{4.19}$$

which can be obtained directly from (4.11).

The elementary Darboux transformation for U, V (i.e., formulae (1.4) with D given by (4.15)) reads

$$\begin{aligned} \tilde{U} &= \frac{M_1}{\lambda - \lambda_1} + \lambda \mathcal{N} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathcal{N}^{-1} - \mathcal{N} \begin{pmatrix} -\alpha & \lambda_1 \\ 1 & \alpha \end{pmatrix} \mathcal{N}^{-1} + \mathcal{N}_{,1}\mathcal{N}^{-1}, \\ \tilde{V} &= \frac{M_2}{\lambda - \lambda_1} + (4\lambda^2 + b\lambda)\mathcal{N} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathcal{N}^{-1} - 4\lambda \mathcal{N} \begin{pmatrix} -\alpha & \lambda_1 \\ 1 & \alpha \end{pmatrix} \mathcal{N}^{-1} + \tilde{V}_0, \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} M_1 &= (u - \lambda_1 - \alpha^2 - \alpha_{,1})\mathcal{N} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathcal{N}^{-1}, \\ M_2 &= \left(-\alpha_{,2} + c + 2a\alpha - b\alpha^2 - 4\lambda_1^2 + 2\lambda_1(u - 2\alpha^2)\right) \mathcal{N} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathcal{N}^{-1} \end{aligned} \tag{4.21}$$

and $\tilde{V}_0$ does not depend on λ (its explicit form follows from lemma 4.2 and, therefore, is automatically preserved by the Darboux transformation).

The necessary condition for the Darboux transformation is vanishing of residues M_1, M_2 (what is equivalent to (4.16) and, as a consequence, to the Riccati equations (4.19)).

In order to assure the Darboux invariance of the coefficients by λ in U and by λ^2 in V we have to impose some constraints on the normalization matrix $\mathcal{N}$ (compare [41]), namely

$$\mathcal{N} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathcal{N}^{-1} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \tag{4.22}$$

what implies the following form of $\mathcal{N}$:

$$\mathcal{N} = f \begin{pmatrix} 0 & 1 \\ -1 & -\gamma \end{pmatrix}, \tag{4.23}$$

where f, γ are functions of x. Now, the transformation (4.20) becomes

$$\begin{aligned} \tilde{U} &= \begin{pmatrix} \gamma - \alpha & 1 \\ \tilde{u} - \lambda & \alpha - \gamma \end{pmatrix} + \frac{f_{,1}}{f} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \tilde{V} &= 4\lambda \begin{pmatrix} \gamma - \alpha & 1 \\ \tilde{v} - \lambda & \alpha - \gamma \end{pmatrix} + \frac{f_{,2}}{f} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \tilde{V}_0, \end{aligned} \tag{4.24}$$

where

$$\tilde{u} = \lambda_1 + 2\gamma\alpha - \gamma^2 - \gamma_{,1}, \qquad \tilde{v} = \lambda_1 + 2\gamma\alpha - \gamma^2 - \tfrac{1}{4}b. \tag{4.25}$$

Comparing (4.24) with (4.13) we find the remaining constraints on the form of the Darboux matrix:

$$f = \text{const}, \qquad \gamma = \alpha, \qquad 2\tilde{v} = \tilde{u}. \tag{4.26}$$

We assume $f = -1$. By virtue of (4.14) $b = 2u$ and we easily verify that the constraint $2\tilde{v} = \tilde{u}$ coincides with the first Riccati equation (4.19).

Corollary 4.3. *The elementary Darboux matrix for the KdV equation is given by*

$$D = \begin{pmatrix} 0 & 1 \\ -1 & -\alpha \end{pmatrix} \begin{pmatrix} \lambda - \lambda_1 & 0 \\ -\alpha & 1 \end{pmatrix} = \begin{pmatrix} -\alpha & 1 \\ \alpha^2 - \lambda + \lambda_1 & -\alpha \end{pmatrix}, \tag{4.27}$$

where α is computed from (4.16), see also (4.18).

The transformation of u can be obtained from (4.19) and (4.25), see (4.28). Taking into account (4.12) we get the transformation for ψ.

Corollary 4.4. *The elementary Darboux matrix (4.27) generates the classical Darboux transformation:*

$$\tilde{\psi} = \psi_{,1} - \alpha\psi \equiv \psi_{,1} - (\ln\hat{\psi}_1)_{,1}\psi, \qquad \tilde{u} = u - 2\alpha_{,1} \equiv u - 2(\ln\hat{\psi}_1)_{,11}, \tag{4.28}$$

where $\hat{\psi}_1 = \hat{\psi}(x, \lambda_1)$ satisfies (4.9).

Formulas (4.28) were first obtained by Darboux [23], see also [46].

Proposition 4.5. *$D \equiv D_{\alpha,\lambda_1}$ given by (4.27) has the following properties: $D^{-1}_{\alpha,\lambda_1}$ is equivalent to $D_{-\alpha,\lambda_1}$ and $D_{\beta,\lambda_2}D_{\alpha,\lambda_1} = \mathcal{N}(\lambda - \lambda_1 + M)$, where $M = (\lambda_1 - \lambda_2)P$ (and $P^2 = P$) for $\lambda_2 \neq \lambda_1$ and $M^2 = 0$ for $\lambda_2 = \lambda_1$.*

Proof. By straightforward computation. First, $D_{-\alpha,\lambda_1} = (\lambda_1 - \lambda)D^{-1}_{\alpha,\lambda_1}$. Then,

$$D_{\beta,\lambda_2}D_{\alpha,\lambda_1} = \begin{pmatrix} -1 & 0 \\ \alpha + \beta & -1 \end{pmatrix} (\lambda - \lambda_1 + M) \tag{4.29}$$

where

$$M = \begin{pmatrix} -\alpha(\alpha + \beta) & \alpha + \beta \\ \alpha(\lambda_1 - \lambda_2) - \alpha^2(\alpha + \beta) & \alpha(\alpha + \beta) - (\lambda_1 - \lambda_2) \end{pmatrix}, \tag{4.30}$$

and we easily verify that $M^2 = (\lambda_1 - \lambda_2)M$, which means that (for $\lambda_2 \neq \lambda_1$) $M = (\lambda_1 - \lambda_2)P$ (where $P^2 = P$), compare (2.5). □

4.4. Nilpotent Darboux matrix and classical binary Darboux transformation

Let us consider the Darboux matrix of the form (2.3). In the case $n = 2$ a nilpotent matrix M ($M^2 = 0$) can be parameterized as

$$M = g \begin{pmatrix} -\sigma & 1 \\ -\sigma^2 & \sigma \end{pmatrix}, \tag{4.31}$$

where g, σ are some functions.

Considering the transformation (1.4) we have to assure that $\tilde{U}_\nu$ are regular at λ_1 by cancelling the pole of second order at $\lambda = \lambda_1$. We get two conditions

$$\begin{aligned} &M_{,\nu} + [M, U_\nu(\lambda_1)] - MU'_\nu(\lambda_1)M = 0, \\ &M_{,\nu}M + MU_\nu(\lambda_1)M = 0, \end{aligned} \tag{4.32}$$

where the prime denotes differentiation with respect to λ. The second set of equations turns out to be a consequence of the first equations (it is enough to multiply them by M from the right).

The classical binary Darboux transformation is usually defined only for the time-independent spectral problem [46]. Therefore, in order to show that the considered transformation (2.3) coincides with the classical binary transformation it is sufficient to confine ourselves to $\nu = 1$. Equations (4.32) (for $\nu = 1$) can be rewritten in terms of g, σ:

$$\begin{aligned} & g_{,1} - 2\sigma g + g^2 = 0, \\ & g_{,1}\sigma + g\sigma_{,1} + \sigma g^2 - g(u - \lambda_1 + \sigma^2) = 0, \\ & g_{,1}\sigma^2 + 2g\sigma\sigma_{,1} + \sigma^2 g^2 - 2g\sigma(u - \lambda_1) = 0. \end{aligned} \tag{4.33}$$

Using the first equation we can reduce the last two equations to

$$\sigma_{,1} + \sigma^2 - u + \lambda_1 = 0. \tag{4.34}$$

Therefore

$$\sigma = \frac{\hat{\psi}_{1,1}}{\hat{\psi}_1}, \tag{4.35}$$

where $\hat{\psi}_1$ satisfies the first equation of (4.9) for $\lambda = \lambda_1$, compare (4.18) and (4.19). Taking into account (4.32) we rewrite (1.4) for $U_1 \equiv U = u_0\lambda + u_1$ as

$$\tilde{u}_0 = \mathcal{N} u_0 \mathcal{N}^{-1}, \qquad \tilde{u}_1 = \mathcal{N}_{,1}\mathcal{N}^{-1} + \mathcal{N}(u_1 + [M, u_0])\mathcal{N}^{-1}. \tag{4.36}$$

In the KdV case, see (4.13), the first equation of (4.36) is satisfied for

$$\mathcal{N} = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}. \tag{4.37}$$

Then, the second equation of (4.36) reduces to

$$\tilde{u} = u + \gamma_{,1} + 2\sigma\gamma + \gamma^2, \qquad g = -\gamma. \tag{4.38}$$

Taking into account the first equation of (4.33) we finally get

$$\tilde{u} = u + 2\gamma_{,1}, \qquad \gamma_{,1} - 2\sigma\gamma - \gamma^2 = 0, \tag{4.39}$$

where σ is given by (4.35). Therefore, the last equation is equivalent to

$$\frac{\partial}{\partial x}\left(\frac{\hat{\psi}_1^2}{\gamma}\right) = -\hat{\psi}_1^2, \tag{4.40}$$

which means that

$$\gamma = \frac{\hat{\psi}_1^2}{c_0 - \int \hat{\psi}_1^2}, \qquad \tilde{u} = u - 2\frac{\partial^2}{\partial x^2}\ln\left|c_0 - \int \hat{\psi}_1^2\right|, \tag{4.41}$$

where c_0 is a constant of integration. The last formula coincides with the classical binary Darboux transformation for the Sturm–Liouville–Schrödinger spectral problem [46].

Corollary 4.6. *The nilpotent Darboux matrix (2.3) generates the classical binary Darboux transformation.*

The 'second' binary Darboux transformation, introduced in [83], corresponds to the choice $c_0 = 1$.

5. Fractional form of the Darboux matrix

Another popular representation of the Darboux matrix (with nondegenerate normalization) is decomposition into partial fractions [56, 79, 80, 82]:

$$
\begin{aligned}
D &= \mathcal{N}\left(I + \frac{A_1}{\lambda - \lambda_1} + \cdots + \frac{A_N}{\lambda - \lambda_N}\right),\\
D^{-1} &= \left(I + \frac{B_1}{\lambda - \mu_1} + \cdots + \frac{B_N}{\lambda - \mu_N}\right)\mathcal{N}^{-1}.
\end{aligned}
\tag{5.1}
$$

In principle the numbers of poles of D and D^{-1} could be different but here, following other papers, we assume the 'symmetric' case (5.1).

We will denote by D_0 the Darboux matrix in the fractional form with the canonical normalization (in other words, $D = \mathcal{N}D_0$). The form (5.1) of D and D^{-1} imposes restrictions on A_k and B_k implied by equations $DD^{-1} = I$ and $D^{-1}D = I$, see (5.2).

Multiplying D by the lowest common multiple of the denominators we obtain the equivalent polynomial form $\hat{D}(\lambda)$ (a polynomial of Nth degree). The determinant $\det \hat{D}(\lambda)$ is a polynomial of degree Nn vanishing at poles of D and D^{-1}, i.e., at $\lambda = \lambda_k$ and $\lambda = \mu_k$ $(k = 1, \ldots, N)$. The sum of multiplicities of all zeros of $\det \hat{D}(\lambda)$ equals Nn. Therefore, for $n = 2$ all zeros are simple, while for $n > 2$ some of them have to be multiple zeros.

The fractional form is convenient in the case of some reductions (e.g., orthogonal or unitary), when the eigenvalues λ_k $(k = 1, \ldots, nN)$ can be naturally divided into pairs λ_k, μ_k.

5.1. Zakharov–Mikhailov's approach

We start from fractional representation of the Darboux matrix (5.1), where A_k, B_k have to satisfy constraints resulting from the condition $DD^{-1} = I$:

$$
\begin{aligned}
&A_k\left(I + \sum_{j=1}^{N}\frac{B_j}{\lambda_k - \mu_j}\right) = 0, \qquad \left(I + \sum_{j=1}^{N}\frac{A_j}{\mu_k - \lambda_j}\right)B_k = 0,\\
&\left(I + \sum_{j=1}^{N}\frac{B_j}{\lambda_k - \mu_j}\right)A_k = 0, \qquad B_k\left(I + \sum_{j=1}^{N}\frac{A_j}{\mu_k - \lambda_j}\right) = 0
\end{aligned}
\tag{5.2}
$$

$(k = 1, \ldots, N)$. We assume the nonisospectral case and demand that $\tilde{U}_\nu$ defined by (1.4) have the same form as U_ν. In particular, it means that the right-hand sides of (1.4) have no poles. Equating to zero the residues at $\lambda = \lambda_j$ and at $\lambda = \mu_k$, we get

$$
\begin{aligned}
&(A_{j,\nu} + A_j U(\lambda_j))\left(I + \sum_{i=1}^{N}\frac{B_i}{\lambda_j - \mu_i}\right) + (L_\nu(\lambda_j) - \lambda_{j,\nu})\sum_{i=1}^{N}\frac{A_j B_i}{(\lambda_j - \mu_i)^2} = 0,\\
&\left(I + \sum_{i=1}^{N}\frac{A_i}{\mu_k - \lambda_i}\right)(U(\mu_k)B_k - B_{k,\nu}) - (L_\nu(\mu_k) - \mu_{k,\nu})\sum_{i=1}^{N}\frac{A_i B_k}{(\mu_k - \lambda_i)^2} = 0,
\end{aligned}
\tag{5.3}
$$

for $j, k = 1, \ldots, N$. Multiplying the first equations by A_j from the right and the second equations by B_k from the left, and then using (5.2), we obtain

$$
(L_\nu(\lambda_j) - \lambda_{j,\nu})\sum_{i=1}^{N}\frac{A_j B_i A_j}{(\lambda_j - \mu_i)^2} = 0, \qquad (L_\nu(\mu_k) - \mu_{k,\nu})\sum_{i=1}^{N}\frac{B_k A_i B_k}{(\mu_k - \lambda_i)^2} = 0,
\tag{5.4}
$$

which is satisfied when (3.4) (and similar equations for μ_k) hold. Note that we derived here a proposition analogous to theorem 3.4.

In order to solve the system (5.2) and (5.3) we assume (3.4) and represent A_k, B_k as follows:

$$A_k = |s_k\rangle\langle a_k|, \qquad B_k = |b_k\rangle\langle q_k| \tag{5.5}$$

where $|s_k\rangle$, $|b_k\rangle$ are matrices built of linearly independent n-component column vectors and $\langle q_k|$, $\langle a_k|$ are matrices built of linearly independent n-component row vectors. In other words, all these matrices have maximal rank. In particular, $|s_k\rangle$ and $\langle a_k|$ have the same rank (denoted by $\mathrm{rk}A_k$) but (in general) different from the rank of $|b_k\rangle$ and $\langle q_k|$ (denoted by $\mathrm{rk}B_k$). Using the notation (5.5) we rewrite equations (5.2) and (5.3) as follows:

$$\begin{aligned} &|s_k\rangle\langle a_k|D_0^{-1}(\lambda_k) = 0, \qquad && D_0(\mu_k)|b_k\rangle\langle p_k| = 0, \\ &D_0^{-1}(\lambda_k)|s_k\rangle\langle a_k| = 0, \qquad && |b_k\rangle\langle p_k|D_0(\mu_k) = 0, \end{aligned} \tag{5.6}$$

$$\begin{aligned} &(|s_k\rangle_{,\nu}\langle a_k| + |s_k\rangle\langle a_k|_{,\nu} + |s_k\rangle\langle a_k|U_\nu(\lambda_k))D_0^{-1}(\lambda_k) = 0, \\ &D_0(\mu_k)(-|b_k\rangle_{,\nu}\langle q_k| - |b_k\rangle\langle q_k|_{,\nu} + |b_k\rangle\langle q_k|U_\nu(\mu_k)) = 0, \end{aligned} \tag{5.7}$$

where $k = 1, \ldots, N$ and $\nu = 1, \ldots, m$. Moreover,

$$D_0(\mu_k) = \left(I + \sum_{j=1}^{N} \frac{|s_j\rangle\langle a_j|}{\mu_k - \lambda_j} \right), \qquad D_0^{-1}(\lambda_k) = \left(I + \sum_{j=1}^{N} \frac{|b_j\rangle\langle q_j|}{\lambda_k - \mu_j} \right). \tag{5.8}$$

Lemma 5.1. *If $|a\rangle$ and $\langle b|$ have the maximal rank, then*

$$|a\rangle\langle b| = 0 \quad \Longleftrightarrow \quad |a\rangle = 0 \quad \text{or} \quad \langle b| = 0. \tag{5.9}$$

Proof. Immediately follows from the definition of the maximal rank. All columns of $|a\rangle$ (and all rows of $\langle b|$) have to be linearly independent. □

Using (5.6) and applying lemma 5.1 to equations (5.7), we get the following linear system:

$$\langle a_k|_{,\nu} = -\langle a_k|U_\nu(\lambda_k), \qquad |b_k\rangle_{,\nu} = U_\nu(\mu_k)|b_k\rangle, \tag{5.10}$$

which is satisfied by

$$\langle a_k| = \langle a_{k0}|\Psi^{-1}(\lambda_k), \qquad |b_k\rangle = \Psi(\mu_k)|b_{k0}\rangle, \tag{5.11}$$

where $\langle a_{k0}|$ and $|b_{k0}\rangle$ are constant. If Ψ is regular at λ_k and μ_k, then the solution given by (5.11) is general (compare section 2.3).

5.2. Symmetric representation of the Darboux matrix

We proceed to derive compact formulae for the remaining ingredients of D, namely for $\langle q_k|$ and $|p_k\rangle$. Taking into account lemma 5.1 we can simplify equations (5.6):

$$\begin{aligned} &\langle a_k| + \sum_{j=1}^{N} M_{kj}\langle q_j| = 0, \qquad && |b_k\rangle - \sum_{j=1}^{N} |s_j\rangle M_{jk} = 0, \\ &\langle q_k| + \sum_{j=1}^{N} K_{kj}\langle a_k| = 0, \qquad && |s_k\rangle - \sum_{j=1}^{N} |b_j\rangle K_{jk} = 0, \end{aligned} \tag{5.12}$$

where

$$M_{kj} = \frac{\langle a_k \mid b_j \rangle}{\lambda_k - \mu_j}, \qquad K_{jk} = \frac{\langle q_j \mid s_k \rangle}{\mu_j - \lambda_k}. \tag{5.13}$$

The expression $\langle a_k \mid b_j \rangle$ denotes matrix multiplication: $\langle a_k \mid b_j \rangle = \langle a_k || b_j \rangle$ (for any fixed j, k). The resulting matrix is not necessarily quadratic. The number of its columns is $\mathrm{rk}(B_j)$ and the number of its rows is $\mathrm{rk}(A_k)$. Similarly, $\langle q_j \mid s_k \rangle$ is also a matrix (for any fixed j, k).

Remark 5.2. Matrices M_{jk} form the so-called soliton correlation matrix $\hat{M}$ which has $\sum_{j=1}^{N} \mathrm{rk}(B_j)$ columns and $\sum_{k=1}^{N} \mathrm{rk}(A_k)$ rows. From (5.12) it follows that

$$\hat{K} = \hat{M}^{-1}. \tag{5.14}$$

Therefore, $\hat{M}$ and $\hat{K}$ have to be quadratic matrices, i.e.,

$$\sum_{k=1}^{N} \mathrm{rk}(A_k) = \sum_{j=1}^{N} \mathrm{rk}(B_j). \tag{5.15}$$

The soliton correlation matrix $\hat{M}$ is a Cauchy-like matrix (compare [53]) which has been reobtained several times in various particular cases (see, for instance, [12, 32, 62, 76]).

Corollary 5.3. *The symmetric form of the multipole Darboux matrix is given by*

$$\begin{aligned} D(\lambda) &= \mathcal{N} \left(I + \sum_{k=1}^{N} \sum_{j=1}^{N} \frac{|b_j\rangle K_{jk} \langle a_k|}{\lambda - \lambda_k} \right), \\ D^{-1}(\lambda) &= \left(I - \sum_{k=1}^{N} \sum_{j=1}^{N} \frac{|b_j\rangle K_{jk} \langle a_k|}{\lambda - \mu_j} \right) \mathcal{N}^{-1}, \end{aligned} \tag{5.16}$$

where $\mathcal{N}$ is a normalization matrix (we assume $\det \mathcal{N} \neq 0$), $\hat{K} = \hat{M}^{-1}$, $\hat{M}$ is given by (5.13), and $|b_j\rangle$, $\langle a_j|$ $(j = 1, \ldots, N)$ are expressed by (5.11).

5.3. How to represent N-soliton surfaces?

Iterated Darboux matrix is a composition of N binary Darboux transformations (see, for instance, [40, 56]):

$$D = \mathcal{N} \left(I + \frac{\lambda_N - \mu_N}{\lambda - \lambda_N} P_N \right) \cdots \left(I + \frac{\lambda_2 - \mu_2}{\lambda - \lambda_2} P_2 \right) \left(I + \frac{\lambda_1 - \mu_1}{\lambda - \lambda_1} P_1 \right), \tag{5.17}$$

where projectors P_k are defined by

$$\ker\, P_k = \Psi_{k-1}(\lambda_k), \qquad \mathrm{im}\, P_k = \Psi_{k-1}(\mu_k), \tag{5.18}$$

where Ψ_k are defined by $\Psi_0(x, \lambda) = \Psi(x, \lambda)$ and (for $k \geqslant 1$):

$$\Psi_k(\lambda) := \left(I + \frac{\lambda_k - \mu_k}{\lambda - \lambda_k} P_k \right) \Psi_{k-1}(\lambda). \tag{5.19}$$

In this case (1.6) yields

$$\tilde{F} = F + \sum_{k=1}^{N} \frac{(\mu_k - \lambda_k)\lambda_{,\zeta}}{(\lambda - \lambda_k)(\lambda - \mu_k)} \Psi_{k-1}^{-1}(\lambda) P_k \Psi_{k-1}(\lambda) \tag{5.20}$$

Note that formula (5.20) does not contain $\mathcal{N}$. Indeed, gauge equivalent linear problems have identical soliton surfaces, see [73].

Remark 5.4. The determinant of the iterated Darboux matrix (5.17) can be easily computed (compare [14]):

$$\det D = \det \mathcal{N} \prod_{k=1}^{N} \left(\frac{\lambda - \lambda_k}{\lambda - \mu_k} \right)^{\dim \operatorname{im} P_k} . \tag{5.21}$$

Usually formula (5.20) is used in the isospectral $SU(n)$ case, when $\lambda_{,\zeta} \equiv 1$, $\mu_k = \bar{\lambda}_k$ (see section 6), and $\operatorname{Tr} F = \operatorname{Tr} \tilde{F} = 0$ (which can be attained by multiplying D by an appropriate factor f, see (1.7)):

$$\tilde{F} = F + \sum_{k=1}^{N} \frac{2\operatorname{Im}\lambda_k}{|\lambda - \lambda_k|} \Psi_{k-1}^{-1}(\lambda) \mathrm{i} \left(\frac{\dim \operatorname{im} P_k}{n} I - P_k \right) \Psi_{k-1}(\lambda) \tag{5.22}$$

(in this case projectors are orthogonal, $P_k^{\dagger} = P_k$, see for example [14, 39, 74]). The sum on the right-hand side of (5.22) consists of traceless components of constant length (using the Killing–Cartan form $a \cdot b = -n \operatorname{Tr}(ab)$ as a scalar product in $su(n)$), see [74]. Thus, formula (5.22) generalizes the classical Bianchi–Lie transformation for pseudospherical surfaces [72, 74].

Formula (5.20) is not manifestly symmetric with respect to permutations of λ_k. The symmetric formula for the Darboux–Bäcklund transformation for soliton surfaces can be obtained by substituting (5.16) into (1.6).

Theorem 5.5. *The symmetric representation for N-soliton surfaces has the form:*

$$\tilde{F} = F - \lambda_{,\zeta} \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\Psi^{-1}(\lambda) |b_j\rangle K_{jk} \langle a_k | \Psi(\lambda)}{(\lambda - \lambda_k)(\lambda - \mu_j)} \tag{5.23}$$

(the notation is explained in corollary 5.3, see also section 1.4).

Proof. We compute $D^{-1} D_{,\lambda}$ where D is given by (5.16):

$$-D^{-1} D_{,\lambda} = \sum_{j,k=1}^{N} \frac{|b_j\rangle K_{jk} \langle a_k|}{(\lambda - \lambda_k)^2} - \sum_{i,j,k,l=1}^{N} \frac{|b_j\rangle K_{ji} \langle a_i \mid b_l\rangle K_{lk} \langle a_k|}{(\lambda - \lambda_k)^2 (\lambda - \mu_j)}.$$

We use (5.13) and perform the summation over i, l in the second component:

$$\sum_{i,l=1}^{N} K_{ji} (\lambda_i - \mu_l) M_{il} K_{lk} = \sum_{i=1}^{N} K_{ji} \lambda_i \hat{\delta}_{ik}^{A} - \sum_{l=1}^{N} \hat{\delta}_{jl}^{B} \mu_l K_{lk} = K_{jk} (\lambda_k - \mu_j), \tag{5.24}$$

where $\hat{\delta}_{ik}^{A}$, $\hat{\delta}_{jl}^{B}$ are natural generalizations of Kronecker's delta (e.g., δ_{kk}^{A} is unit matrix of order $\operatorname{rk}(A_k)$ and δ_{jj}^{B} is unit matrix of order $\operatorname{rk}(B_j)$). Finally, by virtue of an obvious identity $(\lambda - \mu_j) - (\lambda_k - \mu_j) = \lambda - \lambda_k$, we get (5.23). □

Expression (5.23) is a generalization of the symmetric formulae for N-soliton surfaces which has been earlier obtained in the $su(2)$-AKNS case ([12], see also [16]).

Proposition 5.6. *The Darboux matrix fD, with D given by (5.16) and f given by*

$$f = \sqrt[n]{\prod_{k=1}^{N} \frac{(\lambda - \lambda_k)^{\operatorname{rk} A_k}}{(\lambda - \mu_k)^{\operatorname{rk} B_k}}} \tag{5.25}$$

transforms traceless F into traceless $\tilde{F}$. What is more, $\det(fD) = \det N$.

Proof. If D is given by (5.16), then, using (5.23), we compute

$$\frac{\operatorname{Tr}(F-\tilde F)}{\lambda_{,\zeta}} = \operatorname{Tr}\left(\sum_{j,k=1}^{N} \frac{\langle a_k \mid b_j\rangle K_{jk}}{(\lambda-\lambda_k)(\lambda-\mu_j)}\right) = \sum_{k=1}^{N}\left(\frac{\operatorname{rk}(A_k)}{\lambda-\lambda_k} - \frac{\operatorname{rk}(B_k)}{\lambda-\mu_k}\right), \tag{5.26}$$

where we took into account that $\sum_{j=1}^{N} M_{kj}K_{jk}$ is the unit matrix of order $\operatorname{rk}(A_k)$ and $\sum_{k=1}^{N} K_{jk}M_{kj}$ is the unit matrix of order $\operatorname{rk}(B_j)$. Multiplying D by a λ-dependent function we can change $\tilde F - F$, by virtue of (1.7). In order to get $\operatorname{Tr}\tilde F = \operatorname{Tr} F$, we have to take f such that the right-hand side of (5.26) equals $n(\ln f)_{,\lambda}$. Hence we get (5.25).

Surprisingly enough, in this way we can compute also the determinant of D given by (5.16). Indeed, from (1.6) we have $\operatorname{Tr}((fD)^{-1}(fD)_{,\zeta}) = 0$ (provided that $\operatorname{Tr}\tilde F = \operatorname{Tr} F$) and then theorem 3.2 implies that $\det(fD)$ does not depend on ζ (and is λ-independent, as well). Therefore we can evaluate $\det(fD)$ at $\lambda=\infty$. Thus we obtain $\det(fD) = \det\mathcal{N}$. □

Let $\tilde\Psi = fD\Psi$, where D is given by (5.16) and f is given by (5.25). For simplicity we assume also $\operatorname{rk}(A_k) = \operatorname{rk}(B_k) = r_k$. Then

$$\tilde F = F + \lambda_{,\zeta}\Psi^{-1}(\lambda)\sum_{j,k=1}^{N}\left(\frac{(\lambda_k-\mu_k)r_k\delta_{jk} - |b_j\rangle K_{jk}\langle a_k|}{(\lambda-\lambda_k)(\lambda-\mu_j)}\right)\Psi(\lambda), \tag{5.27}$$

where δ_{jk} is Kronecker's delta. Note that the symmetric form of (5.22) is given by the specialization of formula (5.27) to the case $\mu_k = \bar\lambda_k$.

Another representation for multisoliton surfaces can be derived from the polynomial representation of D. In order to compute N-soliton addition to the surface $F := \Psi^{-1}\Psi_{,\lambda}|_{\lambda=\lambda_0}$ we assume the Darboux matrix in a general form

$$D = \sum_{k=0}^{N} T_k(\lambda-\lambda_0)^k.$$

The matrices $T_1,\ldots,T_N$ are computed from the following linear system:

$$\sum_{k=0}^{N} T_k(\lambda_\nu-\lambda_0)^k\Psi(\lambda_\nu)p_\nu = 0, \qquad (\nu = 1,\ldots,Nn),$$

where $\lambda_\nu \in \mathbf{C}$ and $p_\nu \in \mathbf{C}^n$ are constants and T_0 is a given normalization matrix. Of course, one should take care of reductions, which can result in some constraints on λ_ν, p_ν and also on T_0, see section 6. Formula (1.6) assumes the form:

$$\tilde F = F + \Psi^{-1}(\lambda)T_0^{-1}T_1\Psi(\lambda) \equiv F + \Psi^{-1}(\lambda)\theta_1\Psi(\lambda), \tag{5.28}$$

where θ_1 is given by equations analogous to (3.16). Note that also in this case $\tilde F$ does not depend on the normalization matrix T_0 (a change of T_0 implies such change of T_1 that θ_1 remains unchanged, compare section 3.3).

6. Group reductions

The so-called reduction group was introduced by Mikhailov [49] and detailed description of various reductions is given in [50, 80], see also [14]. Group reductions (under a different name) found a rigorous treatment in the framework of the loop group theory [29, 59, 76], compare section 7.3.

In this section we describe several important types of reduction groups. We consider only the case of non-degenerate normalization $\det\mathcal{N} \neq 0$ (which means that $\det\hat D(\lambda)$ is a polynomial of degree Nn). The most convenient form of the Darboux matrix depends on the reduction. The polynomial form (3.1) is very good for reductions to twisted groups, the fractional form (5.1) (and especially its symmetric version (5.16)) is appropriate for unitary and orthogonal reductions. Then, as an example, we present in more detail the principal chiral model (sigma model) and its reductions. The symmetric form (5.16) is of great advantage in this case.

6.1. Reductions to twisted loop groups

Twisted loop groups are defined by $\Psi(\omega\lambda) = Q\Psi(\lambda)Q^{-1}$ where $\omega = \exp\frac{2\pi \mathrm{i}}{K}$ (hence $\omega^K = 1$) and, necessarily, $Q^K = I$ (we assume also $Q = \text{const}$), compare [29]. An important example, two-dimensional Toda chain (then $K = n$), is discussed in detail in [50], using the fractional representation of D (5.1). Here we present a different approach, based on the polynomial representation.

Usually it is better to consider some natural extensions of loop groups which follow from the form of the linear problem. The starting point is the assumption about the form of the linear problem (i.e., U_ν are constrained to the corresponding Lie algebra):

$$U_\nu(\omega\lambda) = QU_\nu(\lambda)Q^{-1}, \tag{6.1}$$

which implies $(\Psi(\omega\lambda))_{,\nu} = U_\nu(\omega\lambda)\Psi(\omega\lambda) = QU_\nu(\lambda)Q^{-1}\Psi(\omega\lambda)$. Hence

$$(Q^{-1}\Psi(\omega\lambda))_{,\nu} = U_\nu(\lambda)(Q^{-1}\Psi(\omega\lambda)),$$

which means (see remark 1.4) that $Q^{-1}\Psi(\omega\lambda) = \Psi(\lambda)C_0(\lambda)$, where the matrix $C_0(\lambda)$ does not depend on x. Therefore

$$\Psi(\omega\lambda) = Q\Psi(\lambda)C_0(\lambda), \tag{6.2}$$

and a similar equation for $\tilde{\Psi} = \hat{D}\Psi$ (with a different $\tilde{C}_0(\lambda)$, in general). Therefore, $\hat{D}(\omega\lambda)Q\Psi(\lambda)C_0(\lambda) = Q\hat{D}(\lambda)\Psi(\lambda)\tilde{C}_0(\lambda)$. In order to eliminate $\Psi(\lambda)$ we have to assume that $\tilde{C}_0(\lambda) = \gamma_0(\lambda)C_0(\lambda)$, where $\gamma_0 : \lambda \to \gamma_0(\lambda) \in \mathbb{C}$ is a rational complex function of λ. Then

$$\begin{aligned} &\hat{D}(\omega\lambda) = \gamma_0(\lambda)Q\hat{D}(\lambda)Q^{-1}, \\ &\det\hat{D}(\omega\lambda) = (\gamma_0(\lambda))^n \det\hat{D}(\lambda). \end{aligned} \tag{6.3}$$

Remark 6.1. Computing $\hat{D}(\omega^2\lambda), \ldots, \hat{D}(\omega^K\lambda)$ we obtain a necessary constraint for γ_0:

$$\gamma_0(\lambda)\gamma_0(\omega\lambda)\cdots\gamma_0(\omega^{K-1}\lambda) = 1. \tag{6.4}$$

This constraint is satisfied by any meromorphic function such that $\gamma_0(\infty) = 1$ and all its zeros and poles coincide with some zeros of $\det\hat{D}(\lambda)$. Note that the matrix $C_0(\lambda)$ also is not arbitrary but satisfies an analogical constraint.

We make usual assumptions: $\gamma_0(\lambda) \equiv 1$ and $C_0(\lambda) \equiv Q^{-1}$ (then Ψ and $\hat{D}$ are fixed points of the reduction group [50], or, in other words, $\hat{D}$ and Ψ take values in the loop group). Then

$$\Psi(\omega^k\lambda) = Q^k\Psi(\lambda)Q^{-k}, \qquad \hat{D}(\omega^k\lambda) = Q^k\hat{D}(\lambda)Q^{-k}, \tag{6.5}$$

for $k = 1, \ldots, K-1$.

Lemma 6.2. *Let* $\gamma_0(\lambda) \equiv 1$. *If* $\det\hat{D}(\lambda_1) = 0$, *then* $\det\hat{D}(\omega^k\lambda_1) = 0$ *for* $k = 1, \ldots, K$. *Multiplicities of all these* K *zeros are identical.*

Therefore, if $\hat{D}(\lambda)$ is a polynomial of order N (and, as a consequence, $\det\hat{D}(\lambda)$ has the order Nn), then Nn has to be divided by K, i.e., there exists an integer $\hat{N}$ such that $Nn = \hat{N}K$

(in the two-dimensional Toda chain case $\hat{N} = N$). Moreover, (6.5) (evaluated at $k = 1$) imply that

$$\omega^N T_0 Q = Q T_0, \tag{6.6}$$

where T_0 is the normalization matrix, compare (3.1). We have to demand that this equation has a solution $T_0 \neq 0$ (otherwise, the Darboux matrix cannot be a polynomial of order N). Certainly (6.6) has a solution for N such that $\omega^N = 1$ (in fact, this assumption was done in [50]).

Corollary 6.3. *If* $\gamma_0(\lambda) \equiv 1$*, then the set of zeros of* $\det \hat{D}(\lambda)$ *is given by* $\{\omega^k \lambda_j | k = 0, 1, \ldots, K-1;\ j = 1, \ldots, \hat{N}\}$*, where* $\lambda_j \in \mathbb{C}$*.*

Equations (3.11), defining the Darboux matrix, can be rewritten as follows (taking into account (6.5)):

$$0 = \hat{D}(\omega^k \lambda_j) \Psi(\omega^k \lambda_j) p_{jk} = Q^k \hat{D}(\lambda_j) \Psi(\lambda_j) Q^{-k} p_{jk}. \tag{6.7}$$

For simplicity we assume the generic case, i.e., all zeros $\omega^k \lambda_j$ are pairwise different (and, as a consequence, simple). Then the kernels of $\hat{D}(\lambda_k)$ are one dimensional, which means that $\Psi(\lambda_j) Q^{-k} p_{jk}$ is proportional to $\Psi(\lambda_j) p_{j0}$.

Theorem 6.4. *Assuming that (6.6) has a solution* $T_0 \neq 0$ *we construct the Darboux matrix (a* λ*-polynomial of order N) according to corollary 3.7 taking into account that its zeros are given by* $\omega^k \lambda_j$ *(see corollary 6.3) and the corresponding eigenvectors are related by* $p_{jk} = Q^k p_{j0}$ *(*$k = 0, 1, \ldots, K-1;\ j = 1, \ldots, \hat{N}$*, where* $\hat{N} = Nn/K$*). This Darboux matrix preserves twisted loop group constraints (6.1) and (6.5), i.e.,* $\tilde{U}_\nu(\omega\lambda) = Q \tilde{U}_\nu(\lambda) Q^{-1}$*, etc.*

Remark 6.5. In the nonisospectral case twisted reductions impose constraints on the form of L_ν. If (3.4) are satisfied, then also $\omega\lambda_{k,\nu} = L_\nu(x, \omega\lambda_k)$. Hence we get the constraint: $\omega L_\nu(x, \lambda) = L_\nu(x, \omega\lambda)$.

The particular case $K = 2$ (i.e., $\omega = -1$) is very popular (e.g., this reduction is necessary to derive the standard linear problem for the famous sine-Gordon equation [56, 60], see also [18]). This case can be generalized by admitting a λ-dependence of Q (actually such generalization can be done for any K but the results have more complicated form, so we omit them). One can easily see that $Q = Q(\lambda)$ has to satisfy

$$Q(-\lambda) Q(\lambda) = \vartheta_0(\lambda) I, \tag{6.8}$$

where ϑ_0 is a scalar function such that $\vartheta_0(-\lambda) = \vartheta_0(\lambda)$ (in particular, we can take $\vartheta_0(\lambda) \equiv 1$). Assuming $\gamma_0 = 1$ we have $\det \hat{D}(-\lambda) = \det \hat{D}(\lambda)$ which means that zeros of $\det \hat{D}(\lambda)$ appear in pairs $\lambda_{k'} = -\lambda_k$. Constant eigenvectors $p_{k'}$ and p_k satisfy $\tilde{\Psi}(\lambda_k) p_k = 0$ and $\tilde{\Psi}(\lambda_{k'}) p_{k'} = 0$ which implies $\tilde{\Psi}(\lambda_k) Q(-\lambda_k) p_{k'} = 0$. If the zero λ_k is simple then $p_{k'} = Q(\lambda_k) p_k$ (the eigenvectors are defined up to a scalar constant factor, therefore we omitted the factor $\vartheta_0(\lambda_k)$). Moreover, the condition (6.6) should be replaced by $\omega^N T_0 Q_\infty = Q_\infty T_0$, where Q_∞ is either $Q(\infty)$ or the coefficient by the highest power of λ in the asymptotic expansion of $Q(\lambda)$ for $\lambda \to \infty$.

6.2. Reality condition

The condition

$$\overline{U_\nu(\lambda)} = U_\nu(\bar{\lambda}) \tag{6.9}$$

(where the bars denote complex conjugates) simply means that all coefficients of matrices U_ν are real. Considering (1.2) we get

$$\overline{\Psi(\bar{\lambda})} = \Psi(\lambda)C(\lambda), \tag{6.10}$$

where $\overline{C(\lambda)}C(\bar{\lambda}) = I$. Applying (6.10) to $\tilde{\Psi}(\lambda) = \hat{D}(\lambda)\Psi(\lambda)$ we obtain the corresponding constraint on $\hat{D}$

$$\overline{\hat{D}(\bar{\lambda})} = \gamma(\lambda)\hat{D}(\lambda), \tag{6.11}$$

where $\gamma(\lambda)$ is a scalar rational function which satisfies $\overline{\gamma(\lambda)}\gamma(\bar{\lambda}) = 1$. Hence

$$(\gamma(\lambda))^n = \frac{\overline{\det \hat{D}(\bar{\lambda})}}{\det \hat{D}(\lambda)}, \tag{6.12}$$

which means that $\gamma = \overline{w(\bar{\lambda})}/w(\lambda)$, where $w(\lambda)$ is a polynomial of degree $K \leqslant N$ (provided that $\hat{D}$ is a polynomial of degree N). Then $\det D(\lambda)$ has K arbitrary zeros of multiplicity n and $(N - K)n$ other zeros which form a set invariant with respect to the complex conjugation, i.e., they are either real or form pairs of conjugate numbers.

We assume the simplest case: $\gamma(\lambda) \equiv 1$ (and also $C(\lambda) = \tilde{C}(\lambda) \equiv 1$) and all zeros of $\det \hat{D}(\lambda)$ are simple. Then either $\lambda_j \in \mathbb{R}$ (then $\bar{p}_j = p_j$) or there are pairs $\lambda_{k'} = \bar{\lambda}_k$ (then $p_{k'} = \bar{p}_k$).

6.3. Unitary reductions

Unitary reductions (which sometimes are also referred to as reality conditions, see for instance [76]) are defined by

$$U^\dagger_{\bar{\nu}}(\bar{\lambda}) = -HU_\nu(\lambda)H^{-1}, \tag{6.13}$$

where the dagger denotes the Hermitean conjugate, H is a constant Hermitean matrix ($H^\dagger = H$) and $\bar{\nu}$ means the complex conjugate (necessary if x^1, x^2 are complex, e.g., usually $x^1 = z, x^2 = \bar{z}$ in the case of chiral models, discussed in section 6.4). Using (6.13) we obtain from (1.2):

$$(\Psi^\dagger(\bar{\lambda}))_{,\nu} \equiv (\Psi(\bar{\lambda})_{,\bar{\nu}})^\dagger = \Psi^\dagger(\bar{\lambda})U^\dagger_{\bar{\nu}}(\bar{\lambda}) \equiv -\Psi^\dagger(\bar{\lambda})H(\lambda)U_\nu(\lambda)H^{-1}(\lambda). \tag{6.14}$$

Taking into account the well-known formula for differentiating the inverse matrix (i.e., $(\Psi^{-1})_{,\nu} = -\Psi^{-1}\Psi_{,\nu}\Psi^{-1}$) we transform (6.14) into

$$H^{-1}((\Psi^\dagger(\bar{\lambda}))^{-1})_{,\nu} = U_\nu(\lambda)H^{-1}(\Psi^\dagger(\bar{\lambda}))^{-1}, \tag{6.15}$$

and, comparing (6.15) with (1.2), we get

$$(\Psi^\dagger(\bar{\lambda}))^{-1} = H\Psi(\lambda)C_0(\lambda), \tag{6.16}$$

where $\Psi(\lambda)$ solves the system (1.2) and $C_0(\lambda)$ is an x-independent matrix. From (6.16) we can derive $C_0^\dagger(\bar{\lambda}) = C_0(\lambda)$. $\tilde{\Psi}(\lambda)$ satisfies the constraint (6.16) with $\tilde{C}_0$ in the place of C_0. Assuming $C_0(\lambda) = k_0(\lambda)\tilde{C}_0(\lambda)$, where $k_0(\lambda)$ is an x-independent scalar function, we derive the condition

$$\hat{D}^\dagger(\bar{\lambda}) = k_0(\lambda)H\hat{D}^{-1}(\lambda)H^{-1}, \tag{6.17}$$

which is necessary for $\hat{D}$ to be the Darboux matrix in the case of unitary reductions. We point out that $k_0(\lambda)$ has to be a rational function. The simplest choice $k_0(\lambda) \equiv 1$ is not possible. Indeed, from (6.17) we obtain

$$(k_0(\lambda))^n = \overline{\det \hat{D}(\bar{\lambda})} \det \hat{D}(\lambda). \tag{6.18}$$

Hence $k_0(\lambda)$ is a polynomial of degree $2N$ (provided that $\det T_0 \neq 0$). Moreover, $\overline{k_0(\bar{\lambda})} = k_0(\lambda)$ which means that $k_0(\lambda)$ is a polynomial with real coefficients. The set of its zeros is symmetric with respect to the real axis. We confine ourselves to the case of $k_0(\lambda)$ without real roots, i.e.

$$k_0(\lambda) = |\det T_0|^2(\lambda - \lambda_1)(\lambda - \bar{\lambda}_1)\cdots(\lambda - \lambda_N)(\lambda - \bar{\lambda}_N)$$

where N is the degree of the polynomial $\hat{D}(\lambda)$. Then

$$\det \hat{D} = (\det T_0)(\lambda - \lambda_1)^{n-d_1}(\lambda - \bar{\lambda}_1)^{d_1}\cdots(\lambda - \lambda_N)^{n-d_N}(\lambda - \bar{\lambda}_N)^{d_N},$$

where d_k are some integers, $1 \leqslant d_k \leqslant n-1$. Note that for $n > 2$ the zeros of $\det \hat{D}$ are, as a rule, degenerate. If λ_k are pairwise different, then dividing $\hat{D}$ by $(\lambda - \lambda_1)\cdots(\lambda - \lambda_N)$ we obtain the matrix D (equivalent to $\hat{D}$) bounded for $\lambda \to \infty$ ($\lim_{\lambda\to\infty} D(\lambda) = T_0$), with singularities at $\lambda = \lambda_k$ ($k = 1, \ldots, N$). Taking into account $\det T_0 \neq 0$, we get

$$D = \frac{\hat{D}(\lambda)}{(\lambda - \lambda_1)\cdots(\lambda - \lambda_N)} = \mathcal{N}\left(I + \sum_{k=1}^{N} \frac{A_k}{\lambda - \lambda_k}\right), \tag{6.19}$$

where $\mathcal{N} = T_0$ and A_k are some matrices dependent on x. The inverse matrix D^{-1} has poles at $\lambda = \bar{\lambda}_k$ ($k = 1, \ldots, N$). Therefore D is exactly of the form (5.1) with $\mu_k = \bar{\lambda}_k$. Note that (6.17) can be rewritten as

$$D^{-1}(\lambda) = H^{-1}D^{\dagger}(\bar{\lambda})H. \tag{6.20}$$

In what follows we use natural notation: $|a^{\dagger}\rangle := (\langle a|)^{\dagger}$, $\langle b^{\dagger}| := (|b\rangle)^{\dagger}$.

Theorem 6.6. *The Darboux matrix of the form (5.16), satisfying the additional constraints*

$$\mu_k = \bar{\lambda}_k, \quad |b_{k0}\rangle = C_0(\bar{\lambda}_k)|a_{k0}^{\dagger}\rangle, \qquad \mathcal{N}^{\dagger}H\mathcal{N} = H, \qquad \mathrm{rk}A_k = \mathrm{rk}B_k, \tag{6.21}$$

($k = 1, \ldots, N$), *preserves the unitary reduction defined by (6.13) and (6.16).*

Proof. We will show that the constraints (6.21), imposed on D given by (5.16), are sufficient to satisfy equation (6.20). The condition $\mu_k = \bar{\lambda}_k$ is already assumed. Equating normalization matrices in (6.20) we obtain $\mathcal{N}^{-1} = H^{-1}\mathcal{N}^{\dagger}H$. The most convenient way to proceed further is to use the symmetric form of the Darboux matrix (5.16). Equating residues at both sides of (6.20) we get $B_k = A_k^{\dagger}$ (compare (5.1)), i.e.,

$$\sum_{j=1}^{N} |b_k\rangle K_{kj}\langle a_j|\mathcal{N}^{-1} = -\sum_{j=1}^{N} H^{-1}\big|a_k^{\dagger}\big\rangle K_{jk}^{\dagger}\big\langle b_j^{\dagger}\big|\mathcal{N}^{\dagger}H. \tag{6.22}$$

In order to satisfy this equation it is sufficient to require

$$\big|a_k^{\dagger}\big\rangle = H|b_k\rangle, \tag{6.23}$$

what implies $\big\langle b_k^{\dagger}\big| = \langle a_k|H^{-1}$. Indeed, using (5.13) we get $M_{jk}^{\dagger} = -M_{kj}$, and, as a consequence, $K_{jk}^{\dagger} = -K_{kj}$, compare (5.14). Finally, (6.23) is implied by (5.11), (6.16) and (6.21). □

Remark 6.7. Assuming $C_0(\lambda) = H^{-1}$ we rewrite the constraint (6.16) as $\Psi^{\dagger}(\bar{\lambda})H\Psi(\lambda) = H$, i.e., $\Psi(\lambda)$ takes values in the same loop group as $D(\lambda)$, compare (6.20). This assumption is not very restrictive. It is sufficient to impose it on initial data (at $x = x_0$). Then it holds for any x.

Remark 6.8. In the non-isospectral case the unitary reduction imposes constraints $\overline{L_{\nu}(x, \bar{\lambda})} = L_{\nu}(x, \lambda)$ on the evolution of λ, compare remark 6.5.

Remark 6.9. By virtue of proposition 5.6 the Darboux–Bäcklund transformation for the reduction $SU(n)$ is generated by the Darboux matrix fD, where D is given by theorem 6.6 and f is given by (5.25) with $\mathrm{rk}B_k = \mathrm{rk}A_k$ and $\mu_k = \bar{\lambda}_k$.

We point out that the case of $k_0(\lambda)$ with real roots is more difficult. Usually, the assumption is made that λ_k are not real, compare [27, 50, 76]. The case of real λ_1, μ_1 is solved and discussed in the case of the binary Darboux matrix ($N = 1$), see [14]. By iterations one can obtain more general solutions. However, it would be interesting to obtain a compact form of the Darboux matrix corresponding to an arbitrary set of eigenvalues (symmetric with respect to the real axis).

6.4. Chiral fields or harmonic maps

As an illustrative example, we will consider the equation

$$(\Phi_{,1}\Phi^{-1})_{,2} + (\Phi_{,2}\Phi^{-1})_{,1} = 0, \tag{6.24}$$

which describes harmonic maps on Lie groups (provided that Φ assumes values in a Lie group G) [76, 77] or, in a physical context, principal chiral fields [30, 79]. Adding another constraint, $\Phi^2 = I$, we get chiral fields (or sigma models) on symmetric (or Grassmann) spaces [1, 31, 61, 79].

The chiral model (6.24) is integrable and the associated isospectral Lax pair $\Psi_{,\nu} = U_\nu\Psi$ is of the form [79]:

$$\Psi_{,1} = \frac{A_1}{1-\lambda}\Psi, \qquad \Psi_{,2} = \frac{A_2}{1+\lambda}\Psi. \tag{6.25}$$

The Lax pairs considered in [27, 77] are equivalent to (6.25) modulo simple transformations of the parameter λ. It is convenient to denote

$$\Phi(x) = \Psi(x, 0). \tag{6.26}$$

Then $A_\nu = \Phi_{,\nu}\Phi^{-1}$ or, in other words,

$$U_\nu(\lambda) = \frac{\Phi_{,\nu}\Phi^{-1}}{1 + (-1)^\nu\lambda}, \tag{6.27}$$

and the compatibility conditions,

$$A_{1,2} + A_{2,1} = 0, \qquad A_{1,2} - A_{2,1} + [A_1, A_2] = 0, \tag{6.28}$$

rewritten in terms of Φ become identical with (6.24).

The Darboux–Bäcklund transformation for Φ (in the case of the principal $GL(n,\mathbb{C})$ chiral model, where there are no restrictions on Φ except non-degeneracy) is given by

$$\tilde{\Phi} = D(0)\Phi, \tag{6.29}$$

where $D(\lambda)$ is represented, for instance, by the symmetric formula (5.16).

The $U(n)$ reduction is defined by the constraint $\Phi^\dagger\Phi = I$ and adding $\det\Phi = 1$ we get $SU(n)$ principal sigma model, see for instance [30, 77]. These constraints are preserved by an appropriately modified Darboux matrix, see theorem 6.6 and remark 6.9.

Chiral models on Grassmann spaces can be characterized by the additional constraint: $\Phi^2 = I$. This is a quite non-trivial reduction, worthwhile to be considered in detail.

Proposition 6.10. *The Lax pair (6.25) satisfies the constraints*

$$U_\nu(\lambda^{-1}) = \Phi_{,\nu}\Phi^{-1} + \Phi U_\nu(\lambda)\Phi^{-1} \qquad (\nu = 1, 2) \tag{6.30}$$

if and only if $\Phi^2 = \mathrm{const}$.

Proof is straightforward. We check that $\Phi U_\nu \Phi^{-1} = -U_\nu$ iff $\Phi^2 = \text{const}$. Then, we compute

$$U_\nu(\lambda^{-1}) = \frac{\lambda \Phi_{,\nu}\Phi^{-1}}{\lambda + (-1)^\nu} = \Phi_{,\nu}\Phi^{-1} - U_\nu(\lambda) = \Phi_{,\nu}\Phi^{-1} + \Phi U_\nu(\lambda)\Phi^{-1},$$

what ends the proof. □

The right-hand side of (6.30) has the form of a gauge transformation. Hence, we immediately have the following conclusions.

Corollary 6.11. *If $\Phi^2 = I$, then*

$$(\Phi^{-1}\Psi(\lambda^{-1}))_{,\nu} = U_\nu(\lambda)\Phi^{-1}\Psi(\lambda^{-1}), \tag{6.31}$$

which means that

$$\Psi(\lambda^{-1}) = \Phi\Psi(\lambda)S_0(\lambda), \tag{6.32}$$

where $S_0(\lambda)$ is a constant matrix. One can check that $S_0(\lambda^{-1}) = S_0^{-1}(\lambda)$.

Proposition 6.12. *The Darboux transformation preserves (6.30) if*

$$D(\lambda^{-1}) = \tilde{\Phi} D(\lambda)\Phi^{-1}, \qquad \tilde{\Phi} = D(0)\Phi. \tag{6.33}$$

Proof. The constraint (6.32) for $\tilde{\Psi} = D\Psi$ reads

$$D(\lambda^{-1})\Psi(\lambda^{-1}) = \tilde{\Phi} D(\lambda)\Psi(\lambda)S_0(\lambda).$$

Using (6.32) we obtain (6.33). Finally, we apply (6.26). □

Corollary 6.13. *Formula (6.33) implies that the divisor of poles of $D(\lambda^{-1})$ has to be exactly the same as the divisor of poles of $D(\lambda)$. Inverting (6.33) we get that divisors of poles of $D^{-1}(\lambda^{-1})$ and $D^{-1}(\lambda)$ also should coincide. Therefore both sets of poles, i.e., $\{\lambda_1, \ldots, \lambda_N\}$ and $\{\mu_1, \ldots, \mu_N\}$, are invariant with respect to the inversion $\lambda \to \lambda^{-1}$.*

Theorem 6.14. *We assume that poles and zeros (λ_k, μ_k) of the Darboux matrix (5.16) can be combined in the following pairs:*

$$\lambda_{k'} = \frac{1}{\lambda_k}, \qquad \mu_{k'} = \frac{1}{\mu_k}, \tag{6.34}$$

and $\lambda_k^2 \neq 1$, $\mu_k^2 \neq 1$. We assume also $\mathcal{N} = I$ and

$$\langle a_{j'0}| = \langle a_{j0}|S_0(\lambda_j), \qquad |b_{j'0}\rangle = S_0^{-1}(\mu_j)|b_{j0}\rangle. \tag{6.35}$$

Under these assumptions the Darboux matrix (5.16) satisfies (6.33) and, moreover,

$$\langle a_{j'}| = \langle a_j|\Phi^{-1}, \qquad |b_{j'}\rangle = \Phi|b_j\rangle. \tag{6.36}$$

Proof. We are going to verify that assumptions of the theorem imply (6.33). First, we will show that assumptions (6.35) imply (6.36). Using (5.11), (6.32), (6.34) and (6.35) we get

$$\begin{aligned} \langle a_{j'}| &= \langle a_{j'0}|\Psi^{-1}(\lambda_{j'}) = \langle a_{j'0}|S_0^{-1}(\lambda_j)\Psi^{-1}(\lambda_j)\Phi^{-1} = \langle a_j|\Phi^{-1}, \\ |b_{j'}\rangle &= \Psi(\mu_{j'})|b_{j'0}\rangle = \Phi\Psi(\mu_j)S_0(\mu_j)|b_{j'0}\rangle = \Phi|b_j\rangle. \end{aligned} \tag{6.37}$$

Then, we compute

$$M_{j'k'} = \frac{\langle a_{j'} \mid b_{k'}\rangle}{\lambda_{j'} - \mu_{k'}} = \frac{\langle a_j \mid b_k\rangle \lambda_j \mu_k}{\mu_k - \lambda_j} = -\lambda_j\mu_k M_{jk}, \tag{6.38}$$

and, vice versa, $M_{jk} = -\lambda_{j'}\mu_{k'}M_{j'k'}$. Hence,

$$K_{k'j'} = -\frac{1}{\mu_k\lambda_j}K_{kj}, \qquad K_{kj} = -\frac{1}{\mu_{k'}\lambda_{j'}}K_{k'j'}. \tag{6.39}$$

Assuming $\mathcal{N} = I$ we proceed to compute ingredients of formula (6.33):

$$D(0) = I - \sum_{j,k=1}^{N} \frac{|b_j\rangle K_{jk}\langle a_k|}{\lambda_k} = I + \sum_{j,k=1}^{N} \frac{\Phi^{-1}|b_j\rangle K_{jk}\langle a_k|\Phi}{\mu_j}, \tag{6.40}$$

where the second equality follows from

$$\sum_{j,k=1}^{N} \frac{|b_j\rangle K_{jk}\langle a_k|}{\lambda_k} = -\sum_{j',k'=1}^{N} \lambda_{k'}\Phi^{-1}|b_{j'}\rangle\left(\frac{K_{j'k'}}{\mu_{j'}\lambda_{k'}}\right)\langle a_{k'}|\Phi \tag{6.41}$$

(primes can be dropped because we sum over the same set of indices). Then, we compute $D(\lambda^{-1})$ and decompose it into the sum of partial fractions:

$$D(\lambda^{-1}) = I - \sum_{j,k=1}^{N} \frac{|b_j\rangle K_{jk}\langle a_k|}{\lambda_k} - \sum_{j,k=1}^{N} \frac{\lambda_k^{-2}|b_j\rangle K_{jk}\langle a_k|}{\lambda - \lambda_k^{-1}}. \tag{6.42}$$

Using (6.40), (6.36) and (6.39), we get (after dropping primes)

$$D(\lambda^{-1}) = D(0) + \sum_{j,k=1}^{N} \frac{\lambda_k\Phi^{-1}|b_j\rangle K_{jk}\langle a_k|\Phi}{\mu_j(\lambda - \lambda_k)}. \tag{6.43}$$

Finally,

$$\tilde{\Phi}D(\lambda)\Phi^{-1} = D(0) + \sum_{j,k=1}^{N} \frac{\Phi|b_j\rangle K_{jk}\langle a_k|\Phi^{-1}}{\lambda - \lambda_k} + \sum_{i,j,k,l=1}^{N} \frac{\Phi^{-1}|b_j\rangle W_{jk}\langle a_k|\Phi^{-1}}{\mu_j(\lambda - \lambda_k)}, \tag{6.44}$$

where

$$W_{jk} = \sum_{i,l=1}^{N} K_{ji}\langle a_i|\Phi^2|b_l\rangle K_{lk} = \sum_{i,l=1}^{N} K_{ji}M_{il}(\lambda_i - \mu_l)K_{lk} = (\lambda_k - \mu_j)K_{jk}, \tag{6.45}$$

where we used $\Phi^2 = I$ and (5.24). Substituting W_{jk} into (6.44) and comparing the result with (6.43) we get (6.33). □

Usually it is sufficient to assume $C_0(\lambda) = H^{-1} = \text{const}$ (compare remark 6.7) and $S_0 = \text{const}$ (but the assumption $S_0 = I$ can be too restrictive).

Proposition 6.15. *We assume* $S_0 = \text{const}$, $H = \text{const}$, $S_0^2 = I$, $H^\dagger = H$ *and* $S_0^\dagger H S_0 = H$. *We consider the Darboux matrix (5.16) such that* $\mathcal{N} = I$, $N = 2K$ *and*

$$\begin{aligned}
&\lambda_{j+K} = \lambda_j^{-1}, \qquad \mu_{j+K} = \mu_j^{-1}, \qquad \mu_k = \bar{\lambda}_k, \qquad |\lambda_j|^2 \neq 1, \qquad \bar{\lambda}_j \neq \lambda_j,\\
&\langle a_{j0}| = \langle b_{j0}^\dagger|H, \qquad \langle a_{j+K,0}| = \langle b_{j+K,0}^\dagger|H,\\
&\langle a_{j+K,0}| = \langle a_{j0}|S_0, \qquad S_0|b_{j+K,0}\rangle = |b_{j0}\rangle,
\end{aligned} \tag{6.46}$$

where $j = 1, \ldots, K$, $k = 1, \ldots, 2K$. *Thus all these data can be expressed by* $\langle a_{10}|, \ldots, \langle a_{K0}|$ *and* $\lambda_1, \ldots, \lambda_K$. *The Darboux–Bäcklund transformation generated by such a Darboux matrix preserves reductions:* $\Psi^\dagger(\bar{\lambda})H\Psi(\lambda) = H$ *and* $\Psi(\lambda^{-1}) = \Psi(0)\Psi(\lambda)S_0$.

Proof. We apply theorems 6.6 and 6.14. It is enough to check whether the equations

$$\langle a_{j0}| = \langle b_{j0}^\dagger|H, \qquad \langle a_{j'0}| = \langle b_{j'0}^\dagger|H, \qquad \langle a_{j'0}| = \langle a_{j0}|S_0, \qquad \langle b_{j0}^\dagger| = \langle b_{j'0}^\dagger|S_0^\dagger$$

are not contradictory. These equations imply $\langle a_{j0}|H^{-1} = \langle a_{j0}|S_0H^{-1}S_0^\dagger$. Hence, using $S_0^2 = I$, we obtain the constraint $S_0^\dagger H S_0 = H$ assuring the compatibility of both reductions. Finally, we denote $j' = j + K$. □

7. Connections with other approaches

In this section we briefly present some other methods of constructing the Darboux–Bäcklund transformation. We show how they are connected with the approach presented in this paper.

7.1. Matrix-valued spectral parameter

The name of Darboux first appeared in the context of the dressing transformations in Matveev's papers (see for instance [45]) who extended the notion of Darboux covariance, known in the case of the Sturm–Liouville–Schrödinger spectral problems, on arbitrary differential operators [46].

In order to apply Matveev's approach to Zakharov–Shabat spectral problems (1.2) the matrix spectral parameter is introduced:

$$\Lambda := \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \tag{7.1}$$

(this notation should not be confused with the function Λ described in remark 1.1). We consider the linear problem of the form [6, 46]:

$$\Psi_{,\nu} = \sum_j \sum_{k=1}^{N_j} U_{\nu kj} \Psi M_j^k + \sum_{k=0}^{N} V_{\nu k} \Psi \Lambda^k, \tag{7.2}$$

where $U_{\nu kj}$ and $V_{\nu k}$ are matrices which do not depend on $\lambda_1, \ldots, \lambda_n$ and

$$M_j := \mathrm{diag}\left(\frac{1}{\lambda_1 - a_j}, \ldots, \frac{1}{\lambda_n - a_j}\right).$$

The following theorem holds [6, 46].

Theorem 7.1. *Equations (7.2) are covariant with respect to the Darboux transformation*

$$\tilde{\Psi} = \Psi\Lambda - \sigma\Psi, \qquad \sigma = \Psi_1 \Lambda_1 \Psi_1^{-1}, \tag{7.3}$$

where Ψ_1 is a fixed solution to (7.2) with Λ replaced by the diagonal matrix $\Lambda_1 = \mathrm{diag}(\lambda_{11}, \ldots, \lambda_{n1})$.

The linear problem (7.2) is closely related to the following special case of the standard Zakharov–Shabat linear problem (1.2):

$$\Phi_{,\nu} = \sum_j \sum_{k=1}^{N_j} U_{\nu kj} \frac{1}{(\lambda - a_j)^k} \Phi + \sum_{k=0}^{N} V_{\nu k} \lambda^k \Phi. \tag{7.4}$$

Namely

$$\Psi(\Lambda) = \{\Phi(\lambda_1)p_1, \ldots, \Phi(\lambda_n)p_n\}, \tag{7.5}$$

where the notation used on the right-hand side (a matrix as a sequence of columns) is the same as in (2.13) and $p_1, \ldots, p_n$ form a constant basis in $\mathbb{C}^n$.

The Darboux matrix generating the transformation (7.3) can be easily computed from theorem 7.1 (using $D = \tilde{\Psi}\Psi^{-1}$). We get

$$D(\Lambda) = \Psi\Lambda\Psi^{-1} - \Psi_1\Lambda_1\Psi_1^{-1}. \tag{7.6}$$

If we put $\lambda_1 = \cdots = \lambda_n = \lambda$, (i.e., $\Lambda = \lambda I$), and $p_k \equiv e_k$ form the canonical basis in $\mathbb{C}^n$ (i.e., $\{p_1, \ldots, p_n\} = I$), then $\Phi(\lambda) = \Psi(\lambda I) \equiv \Psi(\lambda)$. In this case we obtain

$$D(\lambda) = \lambda I - \Psi_1\Lambda_1\Psi_1^{-1}, \tag{7.7}$$

which is the starting point for the construction of the Darboux matrix by Gu and his collaborators [26–28, 84]. Sometimes another form is used:

$$D = I - \lambda\Psi_1\Lambda_1^{-1}\Psi_1^{-1}, \tag{7.8}$$

which is equivalent to (7.7) after changing $\lambda \to \lambda^{-1}$.

7.2. Transfer matrix form of the Darboux matrix

A rational $n \times n$ matrix function $D(\lambda)$, analytic at infinity, can be represented in the form [3, 25]:

$$D(\lambda) = \mathcal{N} + F(\lambda I_N - A)^{-1} G, \tag{7.9}$$

where A is an $N \times N$ matrix, I_N is the unit matrix of order N and $\mathcal{N}$, F, G are matrices of sizes $n \times n$, $n \times N$ and $N \times n$, respectively. Such a representation is called a 'realization' or a 'transfer matrix representation' of D and the number N (i.e., the order of A) is known as the 'state space dimension' of the realization. Realizations are not unique and can have different values of the number N. 'Minimal realizations' have minimal value of N (and the minimal N is called the McMillan degree of D). Minimal realizations are unique up to a change of the basis in the state space (i.e., $F \to FT^{-1}$, $A \to TAT^{-1}$ and $G \to TG$, for some invertible $N \times N$ matrix T) [3, 25].

Proposition 7.2. *If (7.9) is a realization for D, then one of realizations for D^{-1} is given by*

$$D^{-1}(\lambda) = \mathcal{N}^{-1} - \mathcal{N}^{-1} F(\lambda I_N - A + G\mathcal{N}^{-1}F)^{-1} G\mathcal{N}^{-1}. \tag{7.10}$$

The realization (7.10) is minimal iff (7.9) is minimal, see [25].

Formula (7.10) can be verified by a simple but non-trivial computation. The obvious identity $(\lambda I_N - A + G\mathcal{N}^{-1}F) - (\lambda I_N - A) = G\mathcal{N}^{-1}F$ is very helpful, compare (7.16).

Assuming $\mathcal{N} = I$ we consider the so-called transfer matrix

$$W_A(x, \lambda) = I_n - \Pi_2^* S^{-1}(A - \lambda I_N)^{-1}\Pi_1, \tag{7.11}$$

where A, S, Π_1, Π_2^* are some matrices (the star denotes a matrix conjugate, but this is not very important at this moment) and, moreover, the following operator identity holds

$$AS - SB = \Pi_1\Pi_2^*. \tag{7.12}$$

Matrices A, B, Π_1, Π_2, S satisfying (7.12) are said to form an S-colligation [64].

The transfer matrix (7.11) can be used to generate solutions to integrable systems by the Darboux–Bäcklund transformation, see [62, 63]. We can make the following identification:

$$\begin{aligned} &S = \hat{M}, \qquad S^{-1} = \hat{K}, \\ &A = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N), \qquad B = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_N) \end{aligned} \tag{7.13}$$

and, finally

$$\Pi_2^* = (|b_1\rangle, |b_2\rangle, \ldots, |b_N\rangle), \qquad \Pi_1 = \begin{pmatrix} \langle a_1| \\ \langle a_2| \\ \vdots \\ \langle a_N| \end{pmatrix}. \tag{7.14}$$

Corollary 7.3. *The symmetric representation of the Darboux matrix (5.16) can be identified with the transfer matrix form (7.11), where A is diagonal. The identity (7.12) coincides with the definition (5.13) of the matrix $\hat{M}$.*

Constant matrices A of more general form correspond to generalizations of (5.1) (multiple poles are allowed).

In order to show a flavor of the transfer matrix technique we present one of the typical results. Note that the proof of proposition 7.4 is similar to some steps in the proof of theorem 5.5.

Proposition 7.4. *We assume the identity (7.12) and $D \equiv w_A$ is given by formula (7.11). Then*

$$D^{-1} = I_n + \Pi_2^*(B - \lambda I_N)^{-1} S^{-1} \Pi_1. \tag{7.15}$$

Proof. We compute

$$(I_n - \Pi_2^* S^{-1}(A - \lambda I_N)^{-1}\Pi_1)(I_n + \Pi_2^*(B - \lambda I_N)^{-1} S^{-1}\Pi_1) = I_n + \Pi_2^* X \Pi_1,$$

where X is $N \times N$ matrix given by

$$X = (B - \lambda)^{-1} S^{-1} - S^{-1}(A - \lambda)^{-1} - S^{-1}(A - \lambda)^{-1}(AS - SB)(B - \lambda)^{-1} S^{-1}.$$

Now, using the obvious identity

$$AS - SB = (A - \lambda)S - S(B - \lambda), \tag{7.16}$$

we decompose the last component of X into the sum of two terms which immediately cancel with the first two components of X. Therefore $X = 0$ which ends the proof. □

Vectorial Darboux transformations constitute one more approach to Darboux transformations, applied mostly in 2 + 1-dimensional case [43, 44]. Although this technique needs no analogue of the Darboux matrix, the Darboux transformation is expressed by a Cauchy-like matrix and an important role is played by operator identities like (7.12). Comparing the results of [44, 62] we conclude that both methods are in very close correspondence (note that the matrix S of [62] corresponds to the matrix Φ of [44]).

7.3. Factorization in loop groups

Given a Lie group G we define the loop group of G as the group of smooth functions $\gamma : S^1 \to G$, where S^1 denotes the unit circle on the complex plane ($|\lambda| = 1$) [29, 59]. An important role in the loop group theory is played by the Birkhoff factorization theorem. The Birkhoff decomposition is closely related to the Riemann–Hilbert problem which provides a rigorous background for the inverse scattering method [56], see also [29].

In general the Birkhoff factorization is not explicit. The explicit cases are closely related to the construction of Darboux matrices [75–77] (and also to the construction of finite gap solutions), compare similar ideas in the soliton theory [35, 36]. The approach based on the so-called cc-ideals is one more link between the loop group theory and the theory of solitons [33, 34].

From a geometrical point of view the Lax pair consists of commuting differential operators and their compatibility can be interpreted as the condition that a one-parameter family of connections is flat:

$$[\partial_1 - U_1(x, \lambda), \partial_2 - U_2(x, \lambda)] = 0 \tag{7.17}$$

(U_1, U_2 are matrices depending on x through some fields, say u). The 'trivialization' E of a solution u is defined as the solution of the system:

$$E_{,\nu} = -EU_\nu, \qquad E(0, \lambda) = I. \tag{7.18}$$

Then $E(x, \lambda)$ is holomorphic for $\lambda \in \mathbb{C}$, see [76]. The function $E(x, \lambda)$ is also referred to as an 'extended solution', an 'extended frame' or simply a 'frame'. Comparing (7.18) with (1.2) we can identify $E = \Psi^{-1}$. Actually, (7.18) is the adjoint of (1.2), see also (2.4).

Theorem 7.5 (Birkhoff)**.** *The multiplication map μ*

$$\mu : L_+(GL(n, \mathbb{C})) \times L_-(GL(n, \mathbb{C})) \to L(GL(n, \mathbb{C}))$$

is a diffeomorphism onto an open dense subset of $L(GL(n, \mathbb{C}))$, where

- $L_+(GL(n, \mathbb{C}))$ *is the group of holomorphic maps* $h_+ : \mathbb{C} \to GL(n, \mathbb{C})$
- $L_-(GL(n, \mathbb{C}))$ *is the group of holomorphic maps* $h_- : \mathcal{O}_\infty \to GL(n, \mathbb{C})$ *such that* $h_-(\infty) = I$*, where* $\mathcal{O}_\infty$ *is a neighborhood of* $\lambda = \infty$*.*
- $L(GL(n, \mathbb{C}))$ *is the group of holomorphic maps from* $\mathcal{O}_\infty \cap \mathbb{C}$ *to* $GL(n, \mathbb{C})$*.*

Corollary 7.6. *Suppose that* $h_- h_+$ *lies in the image of* μ*. Then, by virtue of the Birkhoff theorem, there exists a unique pair* $f_\pm \in L_\pm(GL(n, \mathbb{C}))$ *such that* $h_- h_+ = f_+ f_-$*. One can interpret it as a 'dressing action' of* h_- *on* h_+ *and* f_+ *is the result of this action, which is denoted by* $h_- \sharp h_+ = f_+$*.*

The dressing action seems to 'forget' about f_-. However, it is worthwhile to stress that this is f_- which should be identified with our Darboux matrix. On the other hand the element h_- is deeply hidden (almost non-existing) in other approaches to the construction of Darboux matrices. In order to explain the dressing action generated by the Birkhoff decomposition we will present the binary Darboux transformation (2.1) in the framework of the loop group approach, following [76].

We assume that $E(x, \lambda) \in L_+(GL(n, \mathbb{C}))$ is given, and we choose the so-called 'simple element' $h_{\lambda_1,\mu_1,\pi} \in L_-(GL(n, \mathbb{C}))$:

$$h_{\lambda_1,\mu_1,\pi}(\lambda) = I + \frac{\lambda_1 - \mu_1}{\lambda - \lambda_1}\pi, \tag{7.19}$$

where λ_1, μ_1 are complex parameters and π is a constant (x-independent) projector in $\mathbb{C}^n$ (i.e., $\pi^2 = \pi$). One can easily see that $h^{-1}_{\lambda_1,\mu_1,\pi} = h_{\mu_1,\lambda_1,\pi}$, compare (2.1) and (2.2).

Then, the Birkhoff theorem states that there exists $\tilde{E} \in L_+(GL(n, \mathbb{C}))$ and $D \in L_-(GL(n, \mathbb{C}))$ such that

$$h_{\lambda_1,\mu_1,\pi} E(x, \lambda) = \tilde{E}(x, \lambda) D(x, \lambda), \tag{7.20}$$

provided that the product on the left-hand side belongs to a certain 'open dense set' of $L(GL(n, \mathbb{C}))$. Now, both the exact form of D and this 'open dense set' can be found by direct calculation. It is sufficient (similarly as in all other approaches discussed earlier) to compare the residues on both sides of equation (7.20). Hence

$$D = I + \frac{\lambda_1 - \mu_1}{\lambda - \lambda_1} P, \tag{7.21}$$

where P is defined by (2.11), where $V_{\rm ker} = \ker \pi$ and $V_{\rm im} = {\rm im}\, \pi$. The open dense set from the Birkhoff theorem is defined by $\ker P \cap {\rm im}\, P = \{0\}$. We remark, by the way, that the Birkhoff theorem assumes the isospectral case and the canonical normalization ($\mathcal{N} = I$).

Note that (7.20) implies $\tilde{E} = hED^{-1} = h\Psi^{-1}D^{-1} = (D\Psi h^{-1})^{-1}$ (where h denotes the simple element). Therefore, $\tilde{\Psi} \equiv \tilde{E}^{-1} = D\Psi h^{-1}$, which is equivalent (because h does not depend on x) to the usual formula $\tilde{\Psi} = D\Psi$ (compare remark 1.4).

8. Invariants of the Darboux transformation

The Darboux transformation changes the matrices U_ν into new matrices $\tilde{U}_k$ of the same form. By invariants of the Darboux transformation we mean constraints on coefficients of U_ν which are preserved by the transformation, see [14] (compare also [69], where one may find many examples). The invariants are very useful in the construction of Darboux matrices in a purely algebraic way, without referring to the special boundary conditions and to the scattering theory (which is usual practice, compare [27, 76, 84]).

Here we simplify the approach of [14] and extend it on the non-isospectral polynomial case. Moreover, we show that our approach works also in a much more general case: when the

Lax pair is singular at some fixed values of the spectral parameter. In this section we denote $U_1 = U$, $U_2 = V$.

8.1. Linear invariants for polynomial Lax pairs

We consider Lax pairs with the following λ-dependence:

$$U = \sum_{k=0}^{\infty} u_k \lambda^{N-k} \equiv \lambda^N u, \qquad V = \sum_{k=0}^{\infty} v_k \lambda^{M-k} \equiv \lambda^M v, \tag{8.1}$$

where N, M are fixed positive integers (not to be confused with the notation of previous sections) and $u_k = u_k(x)$, $v_k = v_k(x)$ $(k = 0, 1, 2, \ldots)$. Usually the sums are finite (i.e., $u_k = v_k = 0$ for sufficiently large k), and this typical case (polynomial in λ and λ^{-1}) corresponds to many classical soliton equations. In particular both U and V can be polynomials in λ (in this case, for $N = 1$, we get the famous AKNS hierarchy). We also assume a similar λ-dependence of the derivatives of λ:

$$\lambda_{,1} = \sum_{k=0}^{\infty} a_k \lambda^{N'-k}, \qquad \lambda_{,2} = \sum_{k=0}^{\infty} b_k \lambda^{M'-k}, \tag{8.2}$$

where N', M' are given integers fixed by the assumption $a_0 \neq 0$, $b_0 \neq 0$ (in the nonisospectral case). The coefficients $a_k = a_k(x)$, $b_k = b_k(x)$ have to satisfy compatibility conditions resulting from $\lambda_{,12} = \lambda_{,21}$ (some examples can be found in [11, 14, 69]).

We consider the Darboux transformation of $U + HV$, where $H = H(x, \lambda)$ is a fixed function

$$H(x, \lambda) = \lambda^{N-M} h(x, \lambda) \equiv \lambda^{N-M}(h_0 + h_1\lambda^{-1} + h_2\lambda^{-2} + \cdots), \tag{8.3}$$

where $h_0, h_1, h_2, \ldots$ are given functions of x. We assume that H is unchanged by the Darboux transformation (and U, V are transformed, as usual, according to (1.4)). The Darboux transformation yields

$$(\tilde{U} + H\tilde{V})D = D_{,1} + HD_{,2} + D(U + HV), \tag{8.4}$$

which reduces to

$$(\tilde{u} + h\tilde{v})D - D(u + hv) = \lambda^{-N} D_{,1} + h\lambda^{-M} D_{,2}. \tag{8.5}$$

We assume that D is analytic at $\lambda = \infty$

$$D = T_0 + T_1\lambda^{-1} + T_2\lambda^{-2} + \cdots, \qquad \det T_0 \neq 0, \tag{8.6}$$

i.e., $D = \lambda^{-N}\hat{D}$, where $\hat{D}$ is given by (3.1).

The idea of linear invariants is quite obvious. Suppose that for $\lambda \approx \infty$ the right-hand side of (8.5) behaves as λ^{-K}, where $K \geqslant 1$. Then the first K terms of the Taylor expansion (in λ^{-1}) of the left-hand side are equal to zero. The first two of these equations read

$$(\tilde{u}_0 + h_0\tilde{v}_0)T_0 = T_0(u_0 + h_0v_0),$$
$$(\tilde{u}_1 + h_0\tilde{v}_1 + h_1\tilde{v}_0)T_0 + (\tilde{u}_0 + h_0\tilde{v}_0)T_1 = T_0(u_1 + h_0v_1 + h_1v_0) + T_1(u_0 + h_0v_0).$$

The assumption $u_0 + h_0v_0 = 0$ implies $\tilde{u}_0 + h_0\tilde{v}_0 = 0$ (provided that $\det T_0 \neq 0$). Then, adding the second assumption: $u_1 + h_0v_1 + h_1v_0 = 0$, we obtain as a consequence $\tilde{u}_1 + h_0\tilde{v}_1 + h_1\tilde{v}_0 = 0$. Thus we have two expressions invariant with respect to the Darboux transformation. Considering the first k (where $k \leqslant K$) equations we get an invariant system of k equations.

We proceed to estimate K. The leading terms of the right-hand side of (8.5) are given by

$$\lambda^{-N}(T_{0,1} - a_0T_1\lambda^{N'-2} + \cdots) + h_0\lambda^{-M}(T_{0,2} - b_0T_1\lambda^{M'-2} + \cdots) \tag{8.7}$$

Therefore $K > k_{\max 1}$, where

$$k_{\max 1} = -1 + \min\{N, M, N + 2 - N', M + 2 - M'\}, \tag{8.8}$$

what can be summarized as follows.

Proposition 8.1. *Suppose that* $0 \leqslant k \leqslant k_{\max 1}$ *and* $h_0, h_1, \ldots, h_k$ *are given functions of* x. *Then the system of* $k + 1$ *linear constraints*

$$u_j + \sum_{i=0}^{j} h_i v_{j-i} = 0, \qquad (j = 0, 1, \ldots, k), \tag{8.9}$$

is invariant with respect to Darboux transformations such that $\det T_0 \neq 0$.

In some special cases, we can formulate stronger propositions (i.e., we have more invariants). In the isospectral case we can replace $k_{\max 1}$ by

$$k'_{\max 1} = -1 + \min\{N, M\}, \tag{8.10}$$

(the same result is valid when $N' \leqslant 2$ and $M' \leqslant 2$). In the case of the canonical normalization ($T_0 = I$) we can replace $k_{\max 1}$ by

$$k''_{\max 1} = \min\{N, M, N + 1 - N', M + 1 - M'\}. \tag{8.11}$$

Below we present one more example.

Proposition 8.2. *Suppose that* $\min\{M, N+2-N', M+2-M'\} > N$ *(it implies, in particular,* $k_{\max 1} = N - 1$*), functions* $h_0, h_1, \ldots, h_k$ $(k \leqslant N)$ *are given, and* T_0 *assume values in some matrix Lie group* G. *Then, the following system of* $k + 1$ *linear constraints is invariant with respect to the Darboux transformation:*

$$\begin{aligned} &u_j + \sum_{i=0}^{j} h_i v_{j-i} = 0, \qquad (j = 0, 1, \ldots, k-1), \\ &u_k + \sum_{i=0}^{k} h_i v_{k-i} \in g, \end{aligned} \tag{8.12}$$

where g *is the Lie algebra of the Lie group* G.

The proof of this proposition is analogous to the proof of proposition 8.1: we consider coefficients by powers of λ^{-1} in (8.5). Only the last step has to be treated in a different way. Assuming that the first k constraints hold, the coefficients by λ^{-k} yield

$$\tilde{u}_k + \sum_{i=0}^{k} h_i \tilde{v}_{k-i} = T_0 \left(u_k + \sum_{i=0}^{k} h_i v_{k-i} \right) T_0^{-1} + \delta_{kN} T_{0,1} T_0^{-1}. \tag{8.13}$$

Now the proof follows immediately from well-known properties of matrix Lie groups ($TgT^{-1} \subset g$ and $T_{,1}T^{-1} \in g$, provided that $T = T(x) \in G$).

8.2. Bilinear invariants for polynomial Lax pairs

Assuming the polynomial form (8.1) of U, V we consider the Darboux transforms of bilinear forms $\mathrm{Tr}(U^2)$, $\mathrm{Tr}(V^2)$ and $\mathrm{Tr}(UV)$. We present computations for the last case (the other two cases are analogous). In this section we use notation: $A \cdot B \equiv \mathrm{Tr}(AB)$. From (1.4) we get

$$\mathrm{Tr}(\tilde{U}\tilde{V}) - \mathrm{Tr}(UV) = \mathrm{Tr}(D_{,1}D^{-1}D_{,2}D^{-1} + D_{,1}VD^{-1} + D_{,2}UD^{-1}). \tag{8.14}$$

The leading terms of the right-hand side of (8.14) read

$$\begin{aligned}
&\lambda^{-(N+M)}\mathrm{Tr}\left((T_{0,1} - a_0 T_1 \lambda^{N'-2} + \cdots)T_0^{-1}(T_{0,2} - b_0 T_1 \lambda^{M'-2} + \cdots)T_0^{-1}\right),\\
&\lambda^{-N}\,\mathrm{Tr}\left((T_{0,1} - a_0 T_1 \lambda^{N'-2} + \cdots)v_0 T_0^{-1}\right),\\
&\lambda^{-M}\,\mathrm{Tr}\left((T_{0,2} - b_0 T_1 \lambda^{M'-2} + \cdots)u_0 T_0^{-1}\right).
\end{aligned} \tag{8.15}$$

Thus the right-hand side of (8.14) behaves as λ^{-K}, where K will be estimated below.

We assume that D and D^{-1} are analytical at $\lambda = \infty$ (i.e., $\det T_0 \neq 0$). Considering coefficients by λ^{-j} $(j = 0, 1, 2, \ldots)$ in formula (8.14), we obtain the following invariants:

$$\begin{aligned}
f_0 &:= u_0 \cdot v_0,\\
f_1 &:= u_0 \cdot v_1 + u_1 \cdot v_0,\\
f_2 &:= u_0 \cdot v_2 + u_1 \cdot v_1 + u_2 \cdot v_0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
f_k &:= u_0 \cdot v_k + u_1 \cdot v_{k-1} + \cdots + u_k \cdot v_0,
\end{aligned} \tag{8.16}$$

where $k < K$. In order to formulate a more precise statement, we define

$$k_{\max 2} = \min\{k_{\max 1}, k_{mn}\}, \tag{8.17}$$

where $k_{mn} = M + N - 1 + \min\{0, 2 - N', 2 - M', 4 - M' - N'\}$ and $k_{\max 1}$ is given by (8.8).

Proposition 8.3. *Bilinear expressions* f_k $(k = 0, \ldots, k_{\max 2})$*, given by (8.16), are preserved by the Darboux transformation (i.e.,* $\tilde{f}_k = f_k$*) provided that* $\det T_0 \neq 0$.

Remark 8.4. If $M > N \geqslant 0$, $N' \leqslant N + 2$, $M' \leqslant M + 2$, then $k_{\max 2} = k_{\max 1}$.

In some cases we can formulate stronger propositions. For $N' \leqslant 2$, $M' \leqslant 2$ (including the isospectral case) $k_{\max 2}$ in proposition 8.3 can be replaced by

$$k'_{\max 2} = -1 + \min\{N, M, N + M\}. \tag{8.18}$$

If the normalization is canonical ($T_0 = I$) we can replace $k_{\max 2}$ by

$$k''_{\max 2} = \min\{k''_{\max 1}, k''_{mn}\}, \tag{8.19}$$

where $k''_{mn} = M + N + \min\{1, 2 - N', 2 - M', 3 - M' - N'\}$.

Analogical considerations can be done for $\mathrm{Tr}\, U^2$ and $\mathrm{Tr}\, V^2$. To obtain the final results (see below) it is enough to substitute $M \to N$, $M' \to N'$ in the first case and $N \to M$, $N' \to M'$ in the second case.

Proposition 8.5. *Suppose that* $0 \leqslant k \leqslant k_{\max 3}$*, where*

$$k_{\max 3} = \min\{N - 1, N + 1 - N', 2N - 1, 2N + 1 - N', 2N + 3 - 2N'\}$$

and $g_0, g_1, \ldots, g_k$ *are given functions of* x*. Then the bilinear constraints*

$$\begin{aligned}
g_0 &:= u_0 \cdot u_0,\\
g_1 &:= u_0 \cdot u_1 + u_1 \cdot u_0,\\
g_2 &:= u_0 \cdot u_2 + u_1 \cdot u_1 + u_2 \cdot u_0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
g_k &:= u_0 \cdot u_k + u_1 \cdot u_{k-1} + \cdots + u_k \cdot u_0
\end{aligned} \tag{8.20}$$

are preserved by the Darboux transformation such that $\det T_0 \neq 0$.

Proposition 8.6. *Suppose that* $0 \leqslant k \leqslant k_{\max 4}$*, where*

$$k_{\max 4} = \min\{M - 1, M + 1 - M', 2M - 1, 2M + 1 - M', 2M + 3 - 2M'\}$$

and $h_0, h_1, \ldots, h_k$ *are given functions of* x*. Then the bilinear constraints*

$$\begin{aligned} h_0 &:= v_0 \cdot v_0, \\ h_1 &:= v_0 \cdot v_1 + v_1 \cdot v_0, \\ h_2 &:= v_0 \cdot v_2 + v_1 \cdot v_1 + v_2 \cdot v_0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ h_k &:= v_0 \cdot v_k + v_1 \cdot v_{k-1} + \cdots + v_k \cdot v_0 \end{aligned} \tag{8.21}$$

are preserved by the Darboux transformation such that $\det T_0 \neq 0$.

8.3. Invariants for general Lax pairs

Let us consider matrices U and V in the neighborhood of $\lambda = \lambda_0$, where U, V have poles of Nth and Mth order, respectively, i.e.,

$$U = \sum_{k=0}^{\infty} u_k (\lambda - \lambda_0)^{k-N}, \qquad V = \sum_{k=0}^{\infty} v_k (\lambda - \lambda_0)^{k-M}, \tag{8.22}$$

where $u_k = u_k(x)$, $v_k = v_k(x)$ $(k = 0, 1, 2, \ldots)$. We are going to show that the general case reduces to the polynomial case discussed above. Indeed, it is sufficient to use the following parameter z in the neighborhood of λ_0:

$$z = (\lambda - \lambda_0)^{-1}, \tag{8.23}$$

and then the Lax pair (8.22) becomes identical with (8.1). Note that $z \to \infty$ for $\lambda \to \lambda_0$. We assume that the Darboux matrix D is analytic at λ_0:

$$D = T_0 + T_1(\lambda - \lambda_0) + T_2(\lambda - \lambda_0)^2 + \cdots = T_0 + z^{-1} T_1 + z^{-2} T_2 + \cdots,$$

where matrices T_k depend on x. In the nonisospectral case we transform the equations (1.3) to the form (8.2):

$$z_{,\nu} = -z^2 L_\nu(x, \lambda_0 + z^{-1}), \tag{8.24}$$

where L_ν have to be expanded in the Laurent (or Taylor) series at $z = \infty$.

In order to obtain linear invariants we consider the linear combination of matrices U, V, given by

$$U + (\lambda - \lambda_0)^{M-N} h V, \tag{8.25}$$

where

$$h = h(x, y; \lambda) \equiv \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k h_k(x, y) = \sum_{k=0}^{\infty} h_k z^{-k} \tag{8.26}$$

is a given scalar function, holomorphic at $\lambda = \lambda_0$. Finally, we arrive at an exact analogue of proposition 8.1.

Bilinear invariants can be treated in the same way. We obtain exact analogues of propositions 8.3, 8.5 and 8.6.

Corollary 8.7. *The polynomial case can be treated as a special subcase, defined by* $\lambda_0 = \infty$*. It is enough to change variables in formulae (8.22):* $\lambda \to \lambda^{-1}$ *(and* $\lambda_0 \to \lambda_0^{-1}$*). Then, making the limit* $\lambda_0 \to 0$*, we get (8.1).*

8.4. Application to the KdV equation

We will show advantages of Darboux invariants considering the case of the KdV equation. Our approach consists in characterizing the Lax pair in terms of some algebraic constraints (see [13, 14]) and then showing that these constraints are preserved by the Darboux–Bäcklund transformation.

Proposition 8.8. *The Lax pair (4.11) can be uniquely characterized by the following set of algebraic constraints:*

(1) U is linear in λ ($U = u_0\lambda + u_1$), $\quad \operatorname{Tr} U = 0$.
(2) V is quadratic in λ ($V = v_0\lambda^2 + v_1\lambda + v_2$), $\quad \operatorname{Tr} V = 0$.
(3) $u_0 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$, $\quad u_1$ is off-diagonal.
(4) $u_0 - \frac{1}{4}v_0 = 0$, $u_1 - \frac{1}{4}v_1 \in g$, $\quad g$ is the one-dimensional Lie algebra spanned by $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.
(5) $u_0 \cdot v_1 + u_1 \cdot v_0 = -8$, $v_1 \cdot v_1 + 2v_0 \cdot v_2 = 0$.
(6) $\overline{U(\lambda)} = U(\bar\lambda)$, $\overline{V(\lambda)} = V(\bar\lambda)$.

Proof. The first four properties imply the following form of U, V:

$$
\begin{aligned}
u_0 &= \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, & u_1 &= \begin{pmatrix} 0 & p \\ u & 0 \end{pmatrix}, & & \\
v_0 &= \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix}, & v_1 &= \begin{pmatrix} 0 & 4p \\ q & 0 \end{pmatrix}, & v_2 &= \begin{pmatrix} -a & b \\ c & a \end{pmatrix},
\end{aligned}
\tag{8.27}
$$

where u, p, q, a, b, c are some complex fields. Bilinear constraints (5) yield

$$
-8p = -8, \qquad 8pq - 8b = 0,
\tag{8.28}
$$

i.e., $p = 1, q = b$. Now compatibility conditions yield the KdV equation (4.10) and expressions (4.14) for a, b, c. The last property implies $u \in \mathbb{R}$. □

The first two constraints are preserved by any Darboux transformation constructed in the standard way, e.g., using corollary 3.7 (and the tracelessness is preserved by virtue of remark 3.3, provided that $\det \mathcal{N} = \text{const}$). The constraints (6) impose restrictions on λ_k and p_k, see section 6.2. In order to preserve the third constraint we have to use freedom in the choice of the normalization matrix $T_0 \equiv \mathcal{N}$. From the first equation of (3.19) we get (taking into account the form of u_0 given by the third constraint)

$$
\mathcal{N} = f \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix},
\tag{8.29}
$$

where f, α are some functions. In the following we put $f = 1$ (thus $\det \mathcal{N} = 1$). Then, denoting $T_1 = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$, we rewrite the second equation of (3.19) as

$$
\begin{pmatrix} 0 & 1 \\ \tilde u & 0 \end{pmatrix} = \begin{pmatrix} -\alpha & 1 \\ u - \alpha^2 & \alpha \end{pmatrix} + \begin{pmatrix} -c_2 & 0 \\ c_1 - \alpha c_2 - c_4 & c_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha_{,1} & 0 \end{pmatrix}.
\tag{8.30}
$$

Hence:

$$
\alpha = -c_2, \qquad \tilde u = u - \alpha^2 - \alpha c_2 + \alpha_{,1} + c_1 - c_4.
\tag{8.31}
$$

The constraint (4) is preserved by virtue of proposition 8.2. Other propositions from section 8 are too weak for our present purposes. Indeed, in the KdV case we have $k_{\max 2} = 0$

and $k_{\max 4} = 1$. Therefore the preservation of the constraints (5) does not follow from propositions 8.3 and 8.6.

Fortunately, the special form of matrices u_0, v_0 and T_0 (given by (8.29) with $f = 1$) for KdV equation allows us to reconsider the behavior of leading terms (8.15). We easily see that any matrix product containing only matrices from the set $\{T_0, T_0^{-1}, T_{0,1}, T_{0,2}, u_0, v_0\}$ (and among them at least one matrix from the set $\{T_{0,1}, T_{0,2}, u_0, v_0\}$) is proportional to u_0, and, as a consequence, it has vanishing trace. Hence, in the case of the KdV equation the Darboux transformation preserves constraints (8.16) for $k = 0, 1$ and constraints (8.21) for $k = 0, 1, 2$. In particular, the constraints (5) of proposition 8.8 are preserved.

Corollary 8.9. *The Darboux transformation (defined as in corollary 3.7) preserves all constraints defining the KdV Lax pair (see proposition 8.8) provided that we impose reality restrictions on* λ_k, p_k *(see section 6.2) and fix the normalization matrix according to formula (8.29) where* $f = 1$ *and* α *is expressed by the matrix* T_1*, namely* $\alpha = -c_2$.

In the case of the elementary Darboux matrix $\det T_0 = 0$ and considerations presented in this section are not applicable. It would be interesting to extend the theory of Darboux invariants in the case $\det T_0 = 0$.

9. Concluding remarks

In this paper we gave a unified view on the Darboux–Bäcklund transformations for 1 + 1-dimensional integrable systems of nonlinear partial differential equations. In particular, we discussed in detail relationships between various approaches to the construction of the Darboux matrix.

Darboux–Bäcklund transformations have been extended in many directions. First of all, they are applicable to 2 + 1-dimensional integrable systems [9, 24, 27, 46, 54], including self-dual Yang–Mills equations [27, 55, 78]. Then, we have 0 + 1-dimensional systems, e.g., ordinary differential equations of nonlinear quantum mechanics [21, 24, 37]. Darboux transformations were also constructed in the supersymmetric case [42, 48] and in the non-commutative case [65].

Matrix representations of spectral problems and Darboux transformations are not always convenient. Impressive examples are associated with Clifford algebras. It is enough to compare the paper [15], where mainly the matrix approach was used, with subsequent papers [5, 17], which are much shorter, more general and more elegant. All these papers consider binary Darboux transformation. An extension on multipole case is not so obvious, compare [20], where some progress in this direction is described. There exist other generalizations of the Darboux transformation on spectral problems with values in abstract associative algebras [10, 24].

The discrete case is (to some extent) very similar to the continuous case. Many aspects (e.g., those concerning the rational dependence on λ and the loop group structure) are just repetitions from the continuous case, compare [24, 39, 45]. It is tempting to apply the ideas of timescales [8], all the more so that in the 'classical' case of the pseudospherical surfaces we succeeded in constructing the Darboux–Bäcklund transformation on arbitrary timescales [19], thus treating the discrete and continuous case in a uniform way. However, some points seem to be more difficult in the discrete case, e.g., Darboux invariants are not formulated yet. Actually, it is not so easy even to find an appropriate discretization of a given integrable system, especially if the associated linear problem is non-isospectral.

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References

[1] Antoine J-P and Piette B 1987 Classical nonlinear σ models on Grassmann manifolds of compact or noncompact type *J. Math. Phys.* **28** 2753–62
[2] Arnold W I 1971 *Ordinary Differential Equations* (Moscow: Nauka) (in Russian)
[3] Bart H, Gohberg I and Kaashoek M A 1979 *Minimal Factorization of Matrix and Operator Functions* (Basle: Birkhäuser)
[4] Belinsky V A and Zakharov V E 1978 Integration of Einstein's equations by means of the inverse scattering technique and construction of exact solutions *Sov. Phys.—JETP* **48** 985–94
[5] Biernacki W and Cieśliński J L 2001 A compact form of the Darboux–Bäcklund transformation for some spectral problems in Clifford algebras *Phys. Lett.* A **288** 167–72
[6] Bobenko A I 1982 Darboux transformations and canonical transformations *Vestnik LGU* **22** 14–21 (in Russian)
[7] Bobenko A I 1994 Surfaces in terms of 2 by 2 matrices. Old and new integrable cases *Harmonic Maps and Integrable Systems (Aspects of Mathematics* vol 23*)* ed A P Fordy and J C Wood (Brunswick: Vieweg) pp 83–128
[8] Bohner M and Peterson A 2001 *Dynamic Equations on Time Scales: An Introduction with Applications* (Boston, MA: Birkhäuser)
[9] Boiti M, Konopelchenko B and Pempinelli F 1985 Bäcklund transformations via gauge transformations in 2 + 1 dimensions *Inverse Problems* **1** 33–56
[10] Boutet de Monvel A and Marchenko V 1994 Generalization of the Darboux transform *Mat. Fiz. Anal. Geom.* **1** 479–504
[11] Burtsev S P, Zakharov V E and Mikhailov A V 1987 Inverse scattering method with variable spectral parameter *Teor. Mat. Fiz.* **70** 323–41 (in Russian)
[12] Cieśliński J 1991 An effective method to compute *N*-soliton Darboux matrix and *N*-soliton surfaces *J. Math. Phys.* **32** 2395–9
[13] Cieśliński J 1995 Algebraic representation of the linear problem as a method to construct the Darboux–Bäcklund transformation *Chaos Solitons Fractals* **5** 2303–13
[14] Cieśliński J 1995 An algebraic method to construct the Darboux matrix *J. Math. Phys.* **36** 5670–706
[15] Cieśliński J 1997 The Darboux–Bianchi transformation for isothermic surfaces. Classical results versus the soliton approach *Diff. Geom. Appl.* **7** 1–28
[16] Cieśliński J 1998 The Darboux–Bianchi–Bäcklund transformation and soliton surfaces *Nonlinearity and Geometry* ed D Wójcik and J Cieśliński (Warsaw: Polish Scientific Publishers PWN) pp 81–107
[17] Cieśliński J L 2000 The Darboux–Bäcklund transformation without using a matrix representation *J. Phys. A: Math. Gen.* **33** L363–8
[18] Cieśliński J L 2003 The structure of spectral problems and geometry: hyperbolic surfaces in E^3 *J. Phys. A: Math. Gen.* **36** 6423–40
[19] Cieśliński J L 2007 Pseudospherical surfaces on time scales: a geometric definition and the spectral approach *J. Phys. A: Math. Theor.* **40** 12525–38
[20] Cieśliński J L and Biernacki W 2005 A new approach to the Darboux–Bäcklund transformation versus the standard dressing method *J. Phys. A: Math. Gen.* **38** 9491–501
[21] Cieśliński J L, Czachor M and Ustinov N V 2003 Darboux covariant equations of von Neumann type and their generalizations *J. Math. Phys.* **44** 1763–80
[22] Coley A, Levi D, Milson R, Rogers C and Winternitz P (ed) 2001 *Bäcklund and Darboux Transformations. The Geometry of Solitons (CRM Proc. Lect. Notes* vol 29*)* (Providence, RI: American Mathematical Society)
[23] Darboux G 1882 Sur une proposition relative aux équations linéaires *C. R. Acad. Sci. (Paris)* **94** 1456–9
[24] Doktorov E V and Leble S B 2007 *A Dressing Method in Mathematical Physics* (Dordrecht: Springer)
[25] Gohberg I, Kaashoek M A and Sakhnovich A L 2002 Scattering problems for a canonical system with a pseudo-exponential potential *Asymptotic Anal.* **29** 1–38
[26] Gu G C 1995 Bäcklund transformations and Darboux transformations *Soliton Theory and Its Applications* ed G C Gu (Berlin: Springer) pp 122–51
[27] Gu G C, Hu H S and Zhou Z X 2005 *Darboux Transformations in Integrable Systems. Theory and their Applications to Geometry* (Dordrecht: Springer)

[28] Gu G C and Zhou Z X 1994 On Darboux transformations for soliton equations in high-dimensional spacetime *Lett. Math. Phys.* **32** 1–10
[29] Guest M A 1997 *Harmonic Maps, Loop Groups, and Integrable Systems* (Cambridge: Cambridge University Press)
[30] Harnad J, Saint-Aubin Y and Shnider S 1984 Superposition of solutions to Bäcklund transformations for SU(n) principal σ-model *J. Math. Phys.* **25** 368–75
[31] Harnad J, Saint-Aubin Y and Shnider S 1984 Bäcklund transformations for nonlinear sigma models with values in Riemannian symmetric spaces *Commun. Math. Phys.* **92** 329–67
[32] Harnad J, Saint-Aubin Y and Shnider S 1984 The soliton correlation matrix and the reduction problem for integrable systems *Commun. Math. Phys.* **93** 33–56
[33] Harrison B K 1983 Unification of Ernst-equation Bäcklund transformations using a modified Wahlquist–Estabrook technique *J. Math. Phys.* **24** 2178–87
[34] Hoenselaers C and Schief W K 1992 Prolongation structures for Harry Dym type equations and Bäcklund transformations of cc-ideals *J. Phys. A: Math. Gen.* **25** 601–22
[35] Its A R 1984 Liouville's theorem and the inverse scattering method *Zapiski Nau. Sem. LOMI* **133** 113–25 (in Russian)
[36] Krichever I M 1977 Methods of algebraic geometry in the theory of non-linear equations *Russ. Math. Surv.* **32** 185–213
[37] Leble S B and Czachor M 1998 Darboux-integrable nonlinear Liouville–von Neumann equation *Phys. Rev.* E **58** 7091–100
[38] Leble S B and Ustinov N V 1993 Deep reductions for matrix Lax system, invariant forms and elementary Darboux transforms *J. Phys. A: Math. Gen.* **26** 5007–16
[39] Levi D, Ragnisco O and Bruschi M 1980 Extension of the Zakharov–Shabat generalized inverse method to solve differential-difference and difference-difference equations *Nuovo Cimento* A **58** 56–66
[40] Levi D, Ragnisco O and Sym A 1984 Dressing method *vs.* classical Darboux transformation *Nuovo Cimento* B **83** 34–41
[41] Levi D, Sym A and Wojciechowski S 1983 A hierarchy of coupled Korteweg–de Vries equations and the normalisation conditions of the Hilbert–Riemann problem *J. Phys. A: Math. Gen.* **16** 2423–32
[42] Liu Q P and Mañas M 1997 Darboux transformations for the Manin–Radul supersymmetric KdV equation *Phys. Lett.* B **394** 337–42
[43] Liu Q P and Mañas M 1999 Vectorial Darboux transformations for the Kadomtsev–Petviashvili hierarchy *J. Nonlinear Sci.* **9** 213–32
[44] Mañas M 1996 Darboux transformations for the nonlinear Schrödinger equations *J. Phys. A: Math. Gen.* **29** 7721–37
[45] Matveev V B 1979 Darboux transformation and the explicit solutions of differential-difference and difference-difference evolution equations: I *Lett. Math. Phys.* **3** 217–22
[46] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)
[47] Meinel R, Neugebauer G and Steudel H 1991 *Solitonen. Nichtlineare Strukturen* (Berlin: Academic) (in German)
[48] Mikhailov A V 1978 Integrability of supersymmetric generalizations of classical chiral models in two-dimensional space-time *Pis. Zh. Eksp. Teor. Fiz* **28** 554–8 (in Russian)
[49] Mikhailov A V 1980 Reductions in integrable systems. The reduction group *Pis. Zh. Eksp. Teor. Fiz* **32** 187–92 (in Russian)
[50] Mikhailov A V 1981 The reduction problem and the inverse scattering method *Physica* D **3** 73–117
[51] Neugebauer G and Kramer D 1983 Einstein–Maxwell solitons *J. Phys. A: Math. Gen.* **16** 1927–36
[52] Neugebauer G and Meinel R 1984 General *N*-soliton solution of the AKNS class on arbitrary background *Phys. Lett.* A **100** 467–70
[53] Nijhoff F, Atkinson J and Hietarinta J 2009 Soliton solutions for ABS lattice equations: I. Cauchy matrix approach *J. Phys. A: Math. Theor.* **42** 404005
[54] Nimmo J J C 1992 Darboux transformations for a two-dimensional Zakharov–Shabat/AKNS spectral problem *Inverse Problems* **8** 219–43
[55] Nimmo J J C, Gilson C R and Ohta Y 2000 Applications of Darboux transformations to the self-dual Yang–Mills equations *Theor. Math. Phys.* **122** 239–46
[56] Novikov S P, Manakov S V, Pitaievsky L P and Zakharov V E 1984 *Theory of Solitons* (New York: Consultants Bureau)
[57] Oevel W and Schief W 1993 Darboux theorems and the KP hierarchy *Applications of Analytic and Geometric Methods to Nonlinear Differential Equations* ed P A Clarkson (Dordrecht: Kluwer) pp 193–206
[58] Park Q H and Shin H J 2001 Darboux transformation and Crum's formula for multi-component integrable equations *Physica* D **157** 1–15

[59] Pressley A N and Segal G B 1986 *Loop Groups* (Oxford: Oxford University Press)

[60] Rogers C and Schief W K 2002 *Bäcklund and Darboux Transformations. Geometry and Modern Applications in Soliton Theory* (Cambridge: Cambridge University Press)

[61] Saint-Aubin Y 1982 Bäcklund transformations and soliton-type solutions for σ models with values in real Grassmannian spaces *Lett. Math. Phys.* **6** 441–7

[62] Sakhnovich A L 1994 Dressing procedure for solutions of nonlinear equations and the method of operator identities *Inverse Problems* **10** 699–710

[63] Sakhnovich A 1996 Iterated Bäcklund–Darboux transformation and transfer matrix-function (nonisospectral case) *Chaos Solitons Fractals* **7** 1251–9

[64] Sakhnovich L A 1986 Factorization problems and operator identities *Usp. Mat. Nauk.* **41** 3–55
Sakhnovich L A 1986 Factorization problems and operator identities *Russ. Math. Surv.* **41** 1–41

[65] Saleem U and Hassan M 2006 Lax pair and Darboux transformation of a noncommutative $U(N)$ principal chiral model *J. Phys. A: Math. Gen.* **39** 11683–96

[66] Sattinger D H and Zurkowski V D 1987 Gauge theory of Bäcklund transformations: II *Physica* D **26** 225–50

[67] Schief W K and Rogers C 1998 Loewner transformations: adjoint and binary Darboux connections *Stud. Appl. Math.* **100** 391–422

[68] Shabat A B 1975 Inverse scattering problem for a system of differential equations *Funk. Anal. Prilozh.* **9** 75–8
Shabat A B 1975 Inverse scattering problem for a system of differential equations *Func. Anal. Appl.* **9** 244–7

[69] Shin H J 2008 Darboux invariants of integrable equations with variable spectral parameters *J. Phys. A: Math. Theor.* **41** 285201

[70] Steudel H, Meinel R and Neugebauer G 1997 Vandermonde-like determinants and *N*-fold Darboux/Bäcklund transformations *J. Math. Phys.* **38** 4692–5

[71] Sym A 1982 Soliton surfaces *Lett. Nuovo Cimento* **33** 394–400

[72] Sym A 1983 Soliton surfaces: II. Geometric unification of solvable nonlinearities *Lett. Nuovo Cimento* **36** 307–12

[73] Sym A 1984 Soliton surfaces: VI. Gauge invariance and final formulation of the approach *Lett. Nuovo Cimento* **41** 353–60

[74] Sym A 1985 Soliton surfaces and their application. Soliton geometry from spectral problems *Geometric Aspects of the Einstein Equations and Integrable Systems (Lecture Notes in Physics* vol 239*)* ed R Martini (Berlin: Springer) pp 154–231

[75] Terng C L 1997 Soliton equations and differential geometry *J. Diff. Geom.* **45** 407–45

[76] Terng C L and Uhlenbeck K 2000 Bäcklund transformations and loop group actions *Commun. Pure Appl. Math.* **53** 1–75

[77] Uhlenbeck K 1989 Harmonic maps into Lie group: classical solutions of the chiral model *J. Diff. Geom.* **30** 1–50

[78] Ustinov N V 1998 The reduced self-dual Yang–Mills equation, binary and infinitesimal Darboux transformations *J. Math. Phys.* **39** 976–85

[79] Zakharov V E and Mikhailov A V 1978 Relativistically invariant systems integrable by the inverse scattering method *Zh. Eksp. Teor. Fiz.* **74** 1953–73 (in Russian)

[80] Zakharov V E and Mikhailov A V 1980 On the integrability of classical spinor models in two-dimensional space-time *Commun. Math. Phys.* **74** 21–40

[81] Zakharov V E and Shabat A B 1974 Integration of nonlinear equations of mathematical physics by the inverse scattering method: I *Funk. Anal. Pril.* **8** 43–53 (in Russian)

[82] Zakharov V E and Shabat A B 1979 Integration of nonlinear equations of mathematical physics by the inverse scattering method: II *Funk. Anal. Pril.* **13** 13–22 (in Russian)

[83] Zeng Y B, Ma W X and Shao Y J 2001 Two binary Darboux transformations for the KdV hierarchy with self-consistent sources *J. Math. Phys.* **42** 2113–28

[84] Zhou Z X 2006 Darboux transformations and exact solutions of two-dimensional $C_l^{(1)}$ and $D_{l+1}^{(2)}$ Toda equations *J. Phys. A: Math. Gen.* **39** 5727–37